

Polynomial Interpolation

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1 Construct a quadratic tracing (1,2), (2,4) and (4,-1). Is this is the only such polynomial?

Following the method shown gives:

$$y = 2 \frac{(x-2)(x-4)}{(1-2)(1-4)} + 4 \frac{(x-1)(x-4)}{(2-1)(2-4)} - \frac{(x-2)(x-1)}{(4-2)(4-1)}$$

And this must be the only such quadratic since 2 quadratics cannot intersect at more than 2 points. This is because $ax^2 + bx^2 + c = 0$ has at most 2 roots.

2 Construct a cubic tracing (1,2), (2,4), (3,7) and (4,-1). Is this the only such polynomial?

As before,

$$y = 2 \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 4 \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + 7 \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} - \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

And this must be the only such cubic since 2 cubics cannot intersect at more than 3 points. This is because $ax^3 + bx^2 + cx + d = 0$ has at most 3 roots.

3 Write an expression for the line passing through (2,7) and (8,-6) in the same way.

$$y = 7 \frac{x-8}{2-8} - 6 \frac{x-2}{8-2}$$

4 Can you always fit a quadratic through the 3 points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) ?

Using the method from before gives:

$$y = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Which is clearly valid except the case in which $x_1 = x_2$, $x_2 = x_3$, $x_1 = x_2$ or $x_1 = x_2 = x_3$

5 Can you always fit a quartic through 5 points (x_i, y_i) ($i = 1, \dots, 5$) where exactly 2 of $x_i = 0$?

Clearly, it is impossible to pass a polynomial through 2 points with equal x -components unless their corresponding y -components are also equal.

6 How many different polynomials can you construct which trace the points (1,1), (2,2), (3,3), (4,4) and (5,5)?

Clearly, $y = x$ is a possible polynomial, and this in fact is what the quartic obtained by the before method simplifies down to.

Another method is to add $(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ since this does not change the output of the function for $x = 1, 2, 4, 5$

And this can be developed further by multiplying this extra part by any polynomial, say $P(x)$. This gives:

$$y = x + (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)P(x)$$

For any polynomial P.

7 Which sets of points is not possible to fit a polynomial?

As in Q5, it is impossible to pass a polynomial through multiple points with equal x -components unless their corresponding y -components are also equal.

8 Extension: consider cases when fitting is or is not unique and when there is no possible fitting.

The most general conclusions I could come to are as follows:

In the case with n points $(x_i, y_i)(i = 1, \dots, n)$ and where $x_1 \neq x_2 \neq \dots \neq x_n$ and where the degree of the polynomial we are constructing is D , it can be broken down into 3 cases:

- $D = n - 1$

In which case we simply use the technique as before, giving

$$y = y_1 \frac{(x - x_2) \dots (x - x_n)}{(x_1 - x_2) \dots (x_1 - x_n)} + \dots + y_n \frac{(x - x_1) \dots (x - x_{n-1})}{(x_n - x_1) \dots (x_n - x_{n-1})}$$

Which is unique since 2 order D polynomials cannot intersect at more than 2 points. This is because $a_D x^D + a_{D-1} x^{D-1} + \dots + a_1 x + a_0 = 0$ has at most D roots.

- $D > n - 1$

By similar reasoning to the previous question, there are infinitely many such polynomials, all of the form:

$$y = y_1 \frac{(x - x_2) \dots (x - x_n)}{(x_1 - x_2) \dots (x_1 - x_n)} + \dots + y_n \frac{(x - x_1) \dots (x - x_{n-1})}{(x_n - x_1) \dots (x_n - x_{n-1})} + (x - x_1) \dots (x - x_n) P(x)$$

For any polynomial P

- $D < n - 1$

To determine whether a set of n points can have a curve of degree $D < n - 1$, choose an arbitrary set of $(D + 1)$ of the total n points. From here use the technique outlined in the first case, namely:

$$y = y_1 \frac{(x - x_2) \dots (x - x_n)}{(x_1 - x_2) \dots (x_1 - x_n)} + \dots + y_n \frac{(x - x_1) \dots (x - x_{n-1})}{(x_n - x_1) \dots (x_n - x_{n-1})}$$

and if this unique curve does not pass all n points, there does not exist a curve as desired. This is because the curve passing the chosen $(D + 1)$ points is unique (by earlier given reasoning), so if there is a curve passing all n points, it is precisely the same curve that passes an arbitrary choice of $(D + 1)$ points.