Areas from Vectors

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The questions to be explored are: Given two vectors in a plane, what is the area of the parallelogram formed? Where else (linear transformations in a plane - a system of 2 linear equations) does this number appear?

Area of a Parallelogram

Firstly, let there be a plane with two orthogonal unit vectors i and j. Thus any vector in the plane can be uniquely expressed as a linear combination of i and j. Let there be two vectors u and v, so that

$$\mathrm{u} = u_1\mathrm{i} + u_2\mathrm{j} \quad \text{and} \quad \mathrm{v} = v_1\mathrm{i} + v_2\mathrm{j}.$$

If

$$\mathrm{i} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \mathrm{j} = \begin{pmatrix}0\\1\end{pmatrix},$$

then the two vectors can be expressed as

$$\mathrm{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix} \quad \text{and} \quad \mathrm{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}.$$

Now that everything is set up, the aim is to find the area of the parallelogram with sides formed by the vectors u and v. If $ABCD$ is a parallelogram and P is the foot of the perpendicular from the point B to AD, then $AD \cdot BP$ is its area. Therefore with the two vectors u and v, the area is $|\mathrm{u}||\mathrm{v}|\sin\alpha$ where α is the acute angle between the two vectors. Define $A(\mathrm{u}, \mathrm{v})$ to be this area, in other words $A(\mathrm{u}, \mathrm{v}) = |\mathrm{u}||\mathrm{v}|\sin(\mathrm{u}, \mathrm{v})$ where (u, v) is the signed acute angle between the vectors u and v such that $\mathrm{v} \to \mathrm{u}$ is the positive direction.

To proceed, a property of this function will be used. Let w, x, y and z be vectors in a plane.

$$A(\mathrm{w}+\mathrm{x}, \mathrm{y}+\mathrm{z}) = A(\mathrm{w}, \mathrm{y}) + A(\mathrm{w}, \mathrm{z}) + A(\mathrm{x}, \mathrm{y}) + A(\mathrm{x}, \mathrm{z})$$

Geometrically, this states that the area between the vectors $\mathrm{w} + \mathrm{x}$ and $\mathrm{y} + \mathrm{z}$ is equal to the sum of the areas between the vectors (w and y), (w and z), (x and y) and (x and z).

Assuming this result (I couldn't prove it) and that $A(p\mathrm{x}, q\mathrm{y}) = pq \cdot A(\mathrm{x}, \mathrm{y})$ which is manifestly true, note that $A(\mathrm{i}, \mathrm{i}) = A(\mathrm{j}, \mathrm{j}) = 0$, $A(\mathrm{i}, \mathrm{j}) = 1$ and $A(\mathrm{j}, \mathrm{i}) = -1$ (angle becomes negative and sine is an odd function). Thus,

$$\begin{aligned} A(\mathrm{u}, \mathrm{v}) &= A(u_1\mathrm{i} + u_2\mathrm{j}, v_1\mathrm{i} + v_2\mathrm{j}) \\ &= u_1v_1A(\mathrm{i}, \mathrm{i}) + u_1v_2A(\mathrm{i}, \mathrm{j}) + u_2v_1A(\mathrm{j}, \mathrm{i}) + u_2v_2A(\mathrm{j}, \mathrm{j}) \\ &= u_1v_2 - v_1u_2. \end{aligned}$$

One thing to note is that this is the *signed* area of the parallelogram. Therefore $|u_1v_2 - v_1u_2|$ is the area.

Determinants

Let there be a transformation T such that for each vector

$$\mathrm{x} = \begin{pmatrix}x\\y\end{pmatrix}, \text{ there is a unique vector } T(\mathrm{x}) = \begin{pmatrix}x'\\y'\end{pmatrix},$$

where x' and y' are defined by

$$\begin{cases} u_1x + v_1y = x' \\ u_2x + v_2y = y'. \end{cases}$$

This transformation can be equivalently defined by $T(\mathrm{x}) = \mathrm{Mx}$, where M is the matrix

$$\mathrm{M} = \begin{pmatrix}u_1 & v_1\\u_2 & v_2\end{pmatrix}.$$

Consider the transformation T on the vertices of a unit square. $T(0) = 0$, $T(\mathrm{i}) = u_1\mathrm{i} + u_2\mathrm{j}$, $T(\mathrm{j}) = v_1\mathrm{i} + v_2\mathrm{j}$ and $T(\mathrm{i}+\mathrm{j}) = (u_1 + v_1)\mathrm{i} + (u_2 + v_2)\mathrm{j}$. It maps the unit square to a parallelogram formed by the vectors u and v whose area was found to be $|u_1v_2 - v_1u_2|$. The number $u_1v_2 - v_1u_2$ is called the *determinant* (of the matrix M or the vectors u and v).

If we solve the equations for x and y, we find the vector x for any image under T. That is, we find another transformation T^{-1} that maps $T(\mathrm{x}) \to \mathrm{x}$, but this is *not always* possible.

$$\begin{cases} u_1x + v_1y = x' \\ u_2x + v_2y = y' \end{cases} \iff \begin{cases} u_2u_1x + u_2v_1y = u_2x' \\ u_1u_2x + u_1v_2y = u_1y' \end{cases}$$

Therefore,

$$x = \frac{v_1y' - v_2x'}{v_1u_2 - v_2u_1} \text{ and } y = \frac{u_2x' - u_1y'}{u_2v_1 - u_1v_2}.$$

Suppose that $u_1v_2 - v_1u_2 = 0$, then the area of the parallelogram is zero, however, it does *not* vanish; it collapses into a line. Thus the transformation maps every vector in the plane to a line. Therefore it is only possible to find a vector x (in fact there will be infinite vectors x) given $T(\mathrm{x})$ if $T(\mathrm{x})$ lies on that line. If it does not, then the system is unsolvable.

To summarise, given a system

$$\begin{cases} u_1x + v_1y = x' \\ u_2x + v_2y = y' \end{cases}$$

there are three possibilities:

(i) The determinant $(u_1v_2 - v_1u_2)$ is non zero and so there is one solution $\left(\frac{v_1y' - v_2x'}{v_1u_2 - v_2u_1}, \frac{u_2x' - u_1y'}{u_2v_1 - u_1v_2}\right)$

(ii) The determinant is zero and there are infinite solutions of the form $(\lambda, \frac{1}{v_2}(y' - u_2\lambda))$

(iii) The determinant is zero and there are no solutions