Combining lengths

Mahdi Raza Khunt
Mahatma Gandhi International School

Tuesday, 28 January 2020

Let a, b, c be the length of the three rods with the condition that $a > b > c$. The total combinations that can be made are listed below out of which only a few are helpful to us.

$$\underbrace{\left.\begin{matrix} + & a & + & b & + & c \\ - & 0 & - & 0 & - & 0 \end{matrix}\right\} 2^6}_{\textit{Choose out of 6}}$$

$+a+b+c$	$+a+b+0$	$+a+b-c$	$+a+b-0$
$+a+0+c$	$+a+0+0$	$+a+0-c$	$+a+0-0$
$+a-b+c$	$+a-b+0$	$+a-b-c$	$+a-b-0$
$+a-0+c$	$+a-0+0$	$+a-0-c$	$+a-0-0$
$+0+b+c$	$+0+b+0$	$+0+b-c$	$+0+b-0$
$+0+0+c$	$+0+0+0$	$+0+0-c$	$+0+0-0$
$+0-b+c$	$+0-b+0$	$+0-b-c$	$+0-b-0$
$+0-0+c$	$+0-0+0$	$+0-0-c$	$+0-0-0$
$-a+b+c$	$-a+b+0$	$-a+b-c$	$-a+b-0$
$-a+0+c$	$-a+0+0$	$-a+0-c$	$-a+0-0$
$-a-b+c$	$-a-b+0$	$-a-b-c$	$-a-b-0$
$-a-0+c$	$-a-0+0$	$-a-0-c$	$-a-0-0$
$-0+b+c$	$-0+b+0$	$-0+b-c$	$-0+b-0$
$-0+0+c$	$-0+0+0$	$-0+0-c$	$-0+0-0$
$-0-b+c$	$-0-b+0$	$-0-b-c$	$-0-b-0$
$-0-0+c$	$-0-0+0$	$-0-0-c$	$-0-0-0$

The terms marked with a box will always be positive. Out of the 2 red underlined terms only one of them will be positive.

$$\text{Either} \quad a > (b + c) \quad \text{or} \quad a < (b + c)$$

Thus the other red term will give a negative value. The rest of the canceled terms are either obtaining a negative value or are repeated.

$\boxed{+a+b+c}$	$\boxed{+a+b+0}$	$\boxed{+a+b-c}$	~~$+a+b-0$~~
$\boxed{+a+0+c}$	$\boxed{+a+0+0}$	$\boxed{+a+0-c}$	~~$+a+0-0$~~
$\boxed{+a-b+c}$	$\boxed{+a-b+0}$	$\underline{+a-b-c}$	~~$+a-b-0$~~
~~$+a-0+c$~~	~~$+a-0+0$~~	~~$+a-0-c$~~	~~$+a-0-0$~~
$\boxed{+0+b+c}$	$\boxed{+0+b+0}$	$\boxed{+0+b-c}$	~~$+0+b-0$~~
$\boxed{+0+0+c}$	~~$+0+0+0$~~	~~$+0+0-c$~~	~~$+0+0-0$~~
~~$+0-b+c$~~	~~$+0-b+0$~~	~~$+0-b-c$~~	~~$+0-b-0$~~
~~$+0-0+c$~~	~~$+0-0+0$~~	~~$+0-0-c$~~	~~$+0-0-0$~~
$\underline{-a+b+c}$	~~$-a+b+0$~~	~~$-a+b-c$~~	~~$-a+b-0$~~
~~$-a+0+c$~~	~~$-a+0+0$~~	~~$-a+0-c$~~	~~$-a+0-0$~~
~~$-a-b+c$~~	~~$-a-b+0$~~	~~$-a-b-c$~~	~~$-a-b-0$~~
~~$-a-0+c$~~	~~$-a-0+0$~~	~~$-a-0-c$~~	~~$-a-0-0$~~
~~$-0+b+c$~~	~~$-0+b+0$~~	~~$-0+b-c$~~	~~$-0+b-0$~~
~~$-0+0+c$~~	~~$-0+0+0$~~	~~$-0+0-c$~~	~~$-0+0-0$~~
~~$-0-b+c$~~	~~$-0-b+0$~~	~~$-0-b-c$~~	~~$-0-b-0$~~
~~$-0-0+c$~~	~~$-0-0+0$~~	~~$-0-0-c$~~	~~$-0-0-0$~~

The above method is a brute force to show that we are not missing out on any combination. But, there is a much simpler way of determining the same thing. We first do this for two given length rods only

Let ‘a’ and ‘b’ be the two length rods we have with a > b. We can make the following combinations

$$a$$
$$b$$
$$a - b$$
$$a + b$$

Introducing another length c < b < a, we can make the following combinations

a	$a + c$	$a - c$
b	$b + c$	$b - c$
$a - b$	$a - b + c$	$\underline{a - b - c}$
$a + b$	$a + b + c$	$a + b - c$
Original Combination	Adding c to Original Combination	Subtracting c from Original Combination

If $b + c > a$ then the term $a - b - c$ will be negative. Then in this case we can make the other term which is $b + c - a$.

And we also forget that c itself can be a length.

We get the total of 11 + 1 + 1 = 13 terms

Thus we can make make at most 13 lengths from 1

Using this information we can determine that using three rods to achieve the maximum distinct rod lengths from 1 to 13, if we can make each term distinct

<u>$1, \quad 3, \quad 9$ is a set of three rods that can be combined to make all rods till length 13 from 1</u>

$$
\begin{aligned}
1 &= 1 \\
2 &= 3 - 1 \\
3 &= 3 \\
4 &= 3 + 1 \\
5 &= 9 - 3 - 1 \\
6 &= 9 - 3 \\
7 &= 9 - 3 + 1 \\
8 &= 9 - 1 \\
9 &= 9 \\
10 &= 9 + 1 \\
11 &= 9 + 3 - 1 \\
12 &= 9 + 3 \\
13 &= 9 + 3 + 1
\end{aligned}
$$

To know if there exists another triple that can make the consecutive number till 13, we go systematically by checking if making length 1 is possible and no combination is giving a repeated result (i.e. all terms are distinct)

There are two cases by how length 1 can be made.

<u>Case 1:</u>

We have a rod length 1
Thus the maximum of a + b + c = 13, hence a + b = 12

$12 = 2 + 10 \implies (1,2,10)$ We can't make length 4
$12 = 3 + 9 \implies (1,3,9)$ ✓
$12 = 4 + 8 \implies (1,4,8)$ We can't make length 2
$12 = 5 + 7 \implies (1,5,7)$ We can make 1 in two ways 7-(5+1)

Only (1,3,9) satisfy, the rest of the pairs defy our condition of a > b.

Case 2:

We don't have 1 in our triple but we use two consecutive rods to make length 1.

$(2,3) \implies (2,3,8) \implies$ We can't make length 4
$(3,4) \implies (3,4,6) \implies$ 7 can be made in two ways, $\quad 7 = 4 + 3, \quad 7 = 6 + 4 - 3$
$(4,5) \implies (4,5,4) \implies$ Lengths have to be distinct
$(5,6) \implies (5,6,2) \implies$ 1 can be made in two ways, $\quad 1 = 6 - 5, \quad 1 = 5 + 2 - 6$
$(6,7) \implies (6,7,0) \implies$ 0 can't be chosen

And for the rest, we need to choose negative values.

<u>Thus, only the triple of (1,3,9) exists</u>

Using a set of four rods:

Let the length of the 4 roads be a, b, c, d and $a > b > c > d$

a	$a + (d)$	$a - (d)$
b	$b + (d)$	$b - (d)$
c	$c + (d)$	$c - (d)$
$a - b$	$a - b + (d)$	$\underline{a - b - (d)}$
$a + b$	$a + b + (d)$	$a + b - (d)$
$a + c$	$a + c + (d)$	$a + c - (d)$
$b + c$	$b + c + (d)$	$b + c - (d)$
$a - b + c$	$a - b + c + (d)$	$a - b + c - (d)$
$a + b + c$	$a + b + c + (d)$	$a + b + c - (d)$
$a - c$	$a - c + (d)$	$\underline{a - c - (d)}$
$b - c$	$b - c + (d)$	$\underline{b - c - (d)}$
$a - b - c \quad / \quad -a + b + c$	$a - b - c + (d) \quad / \quad -a + b + c + (d)$	$\underline{a - b - c - (d) \quad / \quad -a + b + c - (d)}$
$a + b - c$	$a + b - c + (d)$	$a + b - c - (d)$

$\underline{a - b - (d)}$, $\underline{a - b - (d)}$, $\underline{b - c - (d)}$ and $\underline{a - b - c - (d) \quad / \quad -a + b + c - (d)}$ can be negative depending on the numbers. If the term is giving a negative length, we try the implied combination since the inequality will then be reversed. Here implies means, that the other term will be used

$a - b - (d) \implies -a + b + d$
$a - c - (d) \implies -a + c + d$
$b - c - (d) \implies -b + c + d$
$a - b - c - (d) \quad / \quad -a + b + c - (d) \implies -a + b + c + d \quad / \quad a - b - c + d$ <u>Repeated</u>

Also we can use 'd' itself as a length

Thus we can make a total of (13 x 3) -1 + 1 = 39

Only addition:

If only addition of lengths is allowed we can remove the terms including subtraction and can also generalise it for n set of rods

Using only two set of rods we can make 3 combinations

$$a$$
$$b$$
$$a + b$$

Introducing another length c, we can make the following combinations

Original Combination	Adding c to Original Combination
a	$a + c$
b	$b + c$
$a + b$	$a + b + c$

And c length itself,

Introducing another length d for set of four rods we can:

Original Combination	Adding c to Original Combination
a	$a + (d)$
b	$b + (d)$
$a + b$	$a + b + (d)$
$a + c$	$a + c + (d)$
$b + c$	$b + c + (d)$
$a + b + c$	$a + b + c + (d)$
c	$c + (d)$

And length d

Let R_n denote the combinations we can make using n set of rods only using addition

Then we know

$$
\begin{aligned}
R_1 &= 1 \\
R_2 &= 3 \\
R_3 &= 7 \\
R_4 &= 15 \\
&. \\
&. \\
&. \\
&. \\
R_n &= 2 \cdot R_{n-1} + 1 \\
&= 2^n - 1
\end{aligned}
$$

Thus we can make $2^n - 1$ combinations of lengths for n set of rods if we are allowed only to add and make lengths

∎