Post-16 Live Problems

Mega Quadratic Equations

Find all real solutions of the equation:

$$(x^2 - 5x + 5)^{(x^2 - 11x + 30)} = 1$$

There are six possible solutions - did you find all six?

- When tackling this problem, I first tried to somehow factorise or simplify the expression, however it did not seem to work out. Then, I tried to graph the function by substituting some values, but I could only find 4 solutions.

- Then, I realised that the mega quadratic equation is just an expression in the form: $x^y = 1$. And so, if I could find all pairs of numbers which satisfy the expression, then I can simply make the quadratic equations in the problem to be equal to those pairs of numbers and then factorise to find the solutions.

- Using the rules of exponentials, there are 3 cases which satisfy $x^y = 1$

  i) $1^y = 1$ → 1 raised to any power equals one.

  ii) $y^0 = 1$, $y \neq 0$ → any non-zero number raised to the power of 0 equals 1.

  iii) $(-1)^{2k} = 1$, k is an integer → negative 1 raised to the power of an even number equals 1.

- Let us apply these rules to the problem:

i) $1^y = 1$ → we make the base equal to 1

$$x^2 - 5x + 5 = 1$$
$$x^2 - 5x + 4 = 0$$
$$(x - 1)(x - 4) = 0$$
$$\underline{x = 1 \text{ and } 4}$$

ii) $y^0 = 1$, $y \neq 0$ → so we set the exponent equal to 0

$$x^2 - 11x + 30 = 0$$
$$(x-5)(x-6) = 0$$
$$\underline{x = 5 \text{ and } 6}$$

* Remember $y \neq 0$ so we need to check that the base is not equal to 0 by substituting our solutions.

a) $5^2 - 5(5) + 5 = 5 \neq 0$ b) $6^2 - 5(6) + 5 = 11 \neq 0$

so our solutions are correct.

iii) $(-1)^{2k} = 1$, k is an integer → so we set the base equal to -1.

$$x^2 - 5x + 5 = -1$$
$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$
$$\underline{x = 2 \text{ and } 3}$$

* Remember that 2k is an even number, so we need to check that the exponent is even for both solutions.

a) $2^2 - 11(2) + 30 = 12 =$ even b) $3^2 - 11(3) + 30 = 6 =$ even

• so the six solutions are: $\underline{1, 2, 3, 4, 5, 6}$

Here are some more questions to think about:

1. Find all solutions to $(x^2 - 7x + 11)^{(x^2 - 13x + 42)} = 1$
How do these solutions compare to the first equation?

- To solve this, we apply the same method used in the first equation.

i) $1^y = 1 \rightarrow$ $x^2 - 7x + 11 = 1$
$(x - 2)(x - 5) = 0$
$x = 2$ and 5

ii) $y^0 = 1$, $y \neq 0 \rightarrow$ $x^2 - 13x + 42 = 0$
$(x - 6)(x - 7) = 0$
$x = 6$ and 7

check base $\neq 0$
$6^2 - 7(6) + 11 = 5 \neq 0$
$7^2 - 7(7) + 11 = 11 \neq 0$

iii) $(-1)^{2k} = 1$, k is an integer

$x^2 - 7x + 11 = -1$
$(x - 3)(x - 4) = 0$
$x = 3$ and 4

check exponent is even $\rightarrow$
$3^2 - 13(3) + 42 = 12 =$ even
$4^2 - 13(4) + 42 = 6 =$ even

Solutions: 2, 3, 4, 5, 6, 7

- The first equation had solutions of positive integers from 1 to 6, while the second had solutions from 2 to 7.

2. Can you find a Mega Quadratic Equation with solutions 3, 4, 5, 6, 7, 8? How about 4, 5, 6, 7, 8, 9?...

- Let us number the solutions obtained from equation 1 (E_1) and equation 2 (E_2) from least to greatest.

	1st	2nd	3rd	4th	5th	6th
E_1	1	2	3	4	5	6
E_2	2	3	4	5	6	7

- Looking back at the three cases where $x^y = 1$, we can see that when we apply the first case $1^y = 1$ we get the 1st and 4th solutions for both equations: 1 and 4 for E_1 and 2 and 5 for E_2.
When we apply the second case $y^0 = 1$, $y \neq 0$ we get the 5th and 6th solutions: 5 and 6 for E_1, 6 and 7 for E_2.
When we apply the third case $(-1)^{2k} = 1$ we get the 2nd and 3rd solutions: 2 and 3 for E_1, 3 and 4 for E_2.

- Using this knowledge, we can find a mega quadratic equation with solutions 3, 4, 5, 6, 7, 8 by working backwards.

1. Using the first case, $1^y = 1$ we can find our base by constructing a quadratic equation that, when made equal to 1, gives the 1st and 4th solutions (3 and 6).

$x = 3$ and 6
$(x-3)(x-6)$
$x^2 - 9x + 18$
$x^2 - 9x + 19 = 1$ → our base equation

2. Using the second case $y^0 = 1,\ y \neq 0$ we can find our exponential by constructing an equation that, when made equal to 0, gives the 5th and 6th solutions (7, 8)

$x = 7$ and 8
$(x-7)(x-8)$
$x^2 - 15x + 56 = 0$ → our exponential equation.

To check our equations are correct, we substitute 7 and 8 into our base as $y \neq 0$:
$7^2 - 9(7) + 19 = 5 \neq 0$ $8^2 - 9(8) + 19 = 11 \neq 0$

3. Using the third case $(-1)^{ex} = 1$ we need to check that when the base is equal to -1, we get the 2nd and 3rd solutions (4, 5)

$x^2 - 9x + 19 = -1$
$x^2 - 9x + 20 = 0$
$(x-4)(x-5) = 0$
$x = 4, 5$

Finally, we check that the exponent is even when substituting solutions 4 and 5

$4^2 - 15(4) + 56 = 12$ = even $5^2 - 15(5) + 56 = 6$ = even

- Both our base and exponent are correct, so our equation is:

$$\underline{(x^2 - 9x + 19)^{(x^2 - 15x + 56)} = 1}$$

- For an equation with solutions 4, 5, 6, 7, 8, 9 and above, we follow the same method. Alternatively we can see that there is a pattern in the coefficients of the base and exponent of our equations.

	Base Equation	Exponent Equation
E_1	$1x^2 - 5x + 5$	$1x^2 - 11x + 30$
E_2	$1x^2 - 7x + 11$	$1x^2 - 13x + 42$
E_3	$1x^2 - 9x + 19$	$1x^2 - 15x + 56$

- The first term is always equal to 1 in both base and exponent.
- The second term always decreases by 2.
  base sequence: $-5, -7, -9, -11$
  exponent sequence: $-11, -13, -15, -17$
- The third term follows the following sequences:
  base sequence: $5 \overset{+6}{,} 11 \overset{+8}{,} 19 \overset{+10}{,} 29$
  exponent sequence: $30 \overset{+12}{,} 42 \overset{+14}{,} 56 \overset{+16}{,} 72$
- It follows that the equation with solutions 4, 5, 6, 7, 8, 9 is:

$$(x^2 - 11x + 29)^{(x^2 - 17x + 72)} = 1$$

3. Can you explain why there are only 4 solutions to $(x^2 - 5x + 5)^{(x^2 - 9)} = 1$?

i) $1^y = 1 \rightarrow x^2 - 5x + 5 = 1$
$(x-1)(x-4) = 0$
$x = 1$ and 4

ii) $y^0 = 1, y \neq 0 \rightarrow x^2 - 9 = 0$
$x = \pm\sqrt{9}$
$x = 2$ and -2
check base $\neq 0$ $\big\}$ $2^2 - 5(2) + 5 = -1 \neq 0$
$(-2)^2 - 5(-2) + 5 = 19 \neq 0$

iii) $(-1)^{2k} = 1 \rightarrow x^2 - 5x + 5 = -1$
$(x-2)(x-3) = 0$
$x = 2$ and 3
check exp is even $\big\}$ $2^2 - 9 = 0$ = even
$(3)^2 - 9 = 5$ = not even

$\therefore$ exponent cannot be even as it does not satisfy $(-1)^{2k} \neq 1$ so 3 is not a solution.

2 is repeated twice so the solutions are: $\underline{-2, 1, 2, 4}$

4. Can you explain why there are only 3 solutions to $(x^2-6x+10)^{(x^2+x-2)}=1$

i) $1^a=1 \rightarrow$ $x^2-6x+10=1$ since this is a perfect square trinomial, there is only 1 solution

$x^2-6x+9=0$

$(x-3)^2=0$

$\underline{x=3}$

ii) $y^0=1,\ y\neq 0 \rightarrow$ $x^2+x-2=0$ check base $\neq 0$ $\left\{ \begin{array}{l} (-1)^2-6(-1)+10=17\neq 0 \\ 2^2-6(2)+10=2\neq 0 \end{array} \right.$

$(x+1)(x-2)=0$

$x=-1$ and 2

iii) $(-1)^{2n}=1 \rightarrow$ $x^2-6x+10=-1 \rightarrow x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(11)}}{2(1)}=\dfrac{6\pm\sqrt{-8}}{2}$

$x^2-6x+11$

There are no real solutions that satisfy $x^2-6x+10=-1$.

Hence, the solutions are: $\underline{-1, 2, 3}$

5. Can you find a Mega Quadratic Equation with exactly 2 solutions? 5 solutions?

- In general, a mega quadratic equation can have a maximum of 6 solution. This is made evident by the fact that, when we evaluate each of the three cases when $x^y=1$, we can only obtain up to 2 solutions for each case: $2\times 3=6$.

- There are several ways to reduce the number of solutions for a mega quadratic equation.
  * Using perfect square trinomials (which only have 1 solution) as a base and/or exponent to reduce the number of possible solutions.
  * Using equations which do not satisfy the conditions of all 3 cases ($1^a=1$, $y^0=1$ and $(-1)^{2n}=1$).
  * Using equations that do not have real solutions when made equal to 1 and/or -1 for the base; and equal to 0 for the exponent.
  * If a solution is repeated more than once.

Mega Quadratic with 2 solutions:

$$(x^2-12x+37)^{(x^2-6x+9)} = 1$$ solutions: 3 and 6.

i) $1^y = 1 \rightarrow$ $x^2-12x+37=1$ becomes a perfect square when made = to 1 so only 1 solution

$x^2-12x+36=0$

$(x-6)^2=0$

$x=6$

ii) $y^0=1,\ y\neq 0 \rightarrow$ $x^2-6x+9=0$ check base $\neq 0$ $\Big\}$ $3^2-12(3)+37=10\neq 0$

$(x-3)^2=0$

$x=3$

iii) $(-1)^{2k}=1 \rightarrow$ $\left.\begin{array}{l} x^2-12x+37=-1 \\ x^2-12x+38 \end{array}\right\}$ $x = \frac{-(-12)\pm\sqrt{(-12)^2-4(1)(38)}}{2(1)} = \frac{12\pm\sqrt{-8}}{2}$

There are no real solutions that satisfy $x^2-12x+37=-1$

Hence only solutions are: 3, 6

Mega Quadratic with 5 solutions:

$$(x^2-13x+41)^{(x^2-15x+54)} = 1$$ solutions 5, 6, 7, 8, 9

i) $1^y=1 \rightarrow$ $x^2-13x+41=1$

$(x-5)(x-8)=0$

$x=5$ and 8

ii) $y^0=1,\ y\neq 0 \rightarrow$ $x^2-15x+54=0$ check base $\neq 0$ $\Big\}$ $6^2-13(6)+41=-1\neq 0$

$(x-6)(x-9)=0$ $9^2-13(9)+41=5\neq 0$

$x=6$ and 9

iii) $(-1)^{2k}=1 \rightarrow$ $x^2-13x+41=-1$ check exponent is even $\Big\}$ $6^2-15(6)+54=0$ even

$(x-6)(x-7)$ $7^2-15(7)+54=-2$ even

$x=6$ and 7

6 is repeated twice, so the solutions are: 5, 6, 7, 8, 9.