Making a Difference: Stage 3* Zach Thompson

I found 12 answers using the numbers 2,5,7,8:

82	85	72	82	78	57	58	87	75	78	85	87
-75	-72	-58	-57	-52	-28	-27	-52	-28	-25	-27	-25
7	**13**	**14**	**25**	**26**	**29**	**31**	**35**	**47**	**53**	**58**	**62**

I knew that that there were no more, as you can only order 4 digits in a maximum number of 4! ways without repetition. I came up with this expression to explain what I did:

$\frac{n!}{d}$ Where n = total number of digits (4 in this case: 2,5,7,8), and d = digits (2 x 2 digit numbers)

So we get: $\frac{4!}{2} = \frac{4 \times 3 \times 2 \times 1}{2} = 12$ possible combinations of paired numbers

If you have only 1 digit, there is only 1 possible order, A, or 1! But if you have 2 digits, you can order these AB or BA, so 2! = 2 ways. If you have 3 digits, you can have ABC, ACB, BAC, BCA, CAB, CBA, or 3! = 6 different ways. The pattern continues, so 4 digits = 4! 5 digits = 5! etc...

You can also look at this another way: say you have 5 objects, if you chose the first one you have 5 ways of positioning it within a sequence. Now you are left with 4 objects, pick another and you will see that there are only 4 ways of positioning it within the sequence (because one space has already been filled)... and so on. This leads to the expression 5! or $5 \times 4 \times 3 \times 2 \times 1$

4! gives 24 possible 4 digit numbers. These then form 2 x 2 digit numbers but because the answer must be positive, 2857 and 5728 actually give the same sum:

57
-28
29

This means that half of the possible 24 combinations are duplicates, as you always have to take the smaller number from the bigger number to get a positive result.

Follow up Questions

My first thoughts were that you couldn't have 2 digits the same, because at some point you would end with a multiple of 10 (which isn't one of the answers). I experimented with 0 and reasoned that the largest digit would have to be 3 (as 42 – 10 = 32, which is not possible), so I tried {0,1,2,3}

There are still only 12 possible combinations of paired numbers, but because I used consecutive numbers several of the combinations actually gave the same answer.

20	12	30	13	31	23	21	30	23	32	31	32
-13	-03	-21	-02	-20	-10	-03	-12	-01	-10	-02	-01
7	**9**	**9**	**11**	**11**	**13**	**18**	**18**	**22**	**22**	**29**	**31**
1320	0312	2130	0213	2031	1023	0321	1230	0123	1032	0231	0132
2013	1203	3021	1302	3120	2310	2103	3012	2301	3210	3102	3201

Any set of 4 consecutive numbers can be described as $\{n, (n+1), (n+2), (n+3)\}$. Therefore the number in the ten's place can also be described as $10(n), 10(n+1), 10(n+2), 10(n+3)$. Algebra explains why consecutive numbers can give similar answers:

Algebra	Simplified	Examples			
$10(n+2)+(n+3)$ $10n+(n+1)$	$11n+23$ $-\ 11n+\ 1$ **22**	23 -01 **22**	34 -12 **22**	45 -23 **22**	56 -34 **22**

Similarly, the reverse is true:

Algebra	Simplified	Examples			
$10(n+3)+(n+2)$ $10(n+1)+\ n$	$11n+32$ $-\ 11n+10$ **22**	32 -10 **22**	43 -21 **22**	54 -32 **22**	65 -43 **22**

If you reverse two digits, the difference is a multiple of 9, so: (32 – 23) = 9 (43 – 34 = 9), (54 – 45 = 9) etc...

I then arranged the non consecutive digits {2,5,7,8}, to follow the same order. This time the algebra is $\{n, (n+3), (n+5), (n+6)\}$

Sum	Algebra	Simplified
78 -25 **53**	$10(n+5)+(n+6)$ $10n+(n+3)$	$11n+56$ $-\ 11n+\ 3$ **53**

Sum	Algebra	Simplified
87 -52 **35**	$10(n+6)+(n+5)$ $10(n+3)+\ n$	$11n+65$ $-\ 11n+30$ **35**

This time the answer is not the same because the digits are not consecutive. However the digits do reverse; 53 becomes 35 and 22 is a palindrome. So my next task was to use a non palindromic answer and follow the same order with {2,5,7,8}:

Sum	Algebra	Simplified
21 -03 **18**	$10(n+2)+(n+1)$ $10n+(n+3)$	$11n+21$ $-\ 11n+\ 3$ **18**
75 -28 **47**	$10(n+5)+(n+3)$ $10n+(n+6)$	$11n+53$ $-\ 11n+\ 6$ **47**

Sum	Algebra	Simplified
30 -12 **18**	$10(n+3)+n$ $10(n+1)+\ (n+2)$	$11n+30$ $-\ 11n+12$ **18**
82 -57 **25**	$10(n+6)+n$ $10(n+3)+\ (n+5)$	$11n+60$ $-\ 11n+35$ **25**

This time the digits do not interchange.

Consecutive numbers give the same answer because 21 + 9 = 30 and 3 + 9 = 12, so (21 – 3) = (30 – 12), but this doesn't happen with the non consecutive numbers: 75 + 7 = 82, and 28 + 29 = 57, so (47 + 7 – 29 = 25).

Originally, I calculated all the possible combinations using several sets of 4 consecutive 1 digit numbers and found that they all gave exactly the same answers. Finally I tried a non consecutive set {1,3,5,7} with a regular gap between the numbers. I found that this set gave exactly the same answers as the original consecutive sets, but increased by a factor of 2 (because the gap between the digits has doubled).

51	35	71	37	73	57	53	71	57	75	73	75
-37	-17	-53	-15	-51	-31	-17	-35	-13	-31	-15	-13
14	**18**	**18**	**22**	**22**	**26**	**36**	**36**	**44**	**44**	**58**	**62**

This seems to say that regular intervals between the digits in a set, give answers that follow a predictable pattern, probably because in this case the difference is now 18 (31 – 13 = 18) rather than 9 (21 – 12 = 9). I think this is because if you reverse digits, you get multiple of 9 as $(10a + b) - (10b + a) = 9(a - b)$

The last question asked why the smallest answer is always 7 and the largest always 31:

Arrange the 4 consecutive digits into 2-2digit numbers that are as close together as possible in value to get 7, and as far apart as possible to get 31. If you follow the pattern (follow the algebra), this will happen every time.

Algebra	Simplified	Examples		
$10(n + 2) + n$	$11n + 20$	86	75	64
$10(n + 1) + (n + 3)$	$- 11n + 13$	-79	-68	-57
	7	**7**	**7**	**7**

And

Algebra	Simplified	Examples		
$10(n + 3) + (n + 2)$	$11n + 32$	98	87	76
$10n + (n + 1)$	$- 11n + 1$	-67	-56	-45
	31	**31**	**31**	**31**