Reversals: Stage 3**

Zach Thompson

Any 2 digit number can be expressed as $10a + b$ because of place value.

Tens 10's	Units 1's
7	2
a	b

Because you can only write a single digit in each column x in the ten's column is worth 10 times as much as x in the units column.

E.g. 72 = (10x7) + (1x2). So if $a = 7, b = 2$, the a in the tens column is worth $10a$

To reverse the digits $10a + b$ becomes $10b + a$ (72 becomes 27)

1) Question: $9\left(\frac{(10a+b)}{2}\right) = 10b + a$

The best way to answer these questions is via algebra. I used letters to represent what Alison and Charlie were doing. The first step was to write the question in algebra, then work out the answers.

Divide by 9 to get: $\frac{10a+b}{2} = \frac{10b+a}{9}$

Multiply by 18(LCM) to get $90a + 9b = 20b + 2a$

LCM = lowest common multiple

Simplify: $88a = 11b$
$a = 1, b = 8$

(1x88) = (8x11)

Substitute: $9\left(\frac{18}{2}\right) = 9^2 = 81$

81 reversed is 18. Alison chose 18

2) Question: $\left(\frac{(10a+b)+1}{2}\right) = 10b + a$

Multiply by 2 to get: $10a + b + 1 = 20b + 2a$

Simplify: $8a + 1 = 19b$
$a = 7, b = 3$

19 is prime and $8 = 2^3$ So I looked for a ratio that would work. (7 x 8) + 1 = (3 x 19)

Substitute: $\left(\frac{73+1}{2}\right) = \frac{74}{2} = 37$

37 reversed is 73. Alison chose 73

3) Question $\left(\frac{(10a+b)-2}{2}\right) = 10b + a$

Multiply by 2 to get: $10a + b - 2 = 20b + 2a$

Simplify: $8a - 2 = 19b$
$a = 5, b = 2$

(5x8) – 2 = (2x19)

Substitute: $\left(\frac{52-2}{2}\right) = \frac{50}{2} = 25$

25 reversed is 52. Charlie chose 52

4) Question: $\left(\frac{(10a+b)-10}{2}\right) = 10b + a$

Multiply by 2 to get: $10a + b - 10 = 20b + 2a$

Simplify: $8a - 10 = 19b$

$a = 6, b = 2$ (6x8) – 10 = (2x19)

Substitute: $\frac{(62-10)}{2} = \frac{52}{2} = 26$ 26 reversed is 62. I chose 62

Extension:

Any three digit number can be written as $100a + 10b + c$ as the digit in the hundreds place is worth 100 times its unit value, and the digit in the tens place is worth 10 times its unit value.

The question asks us ensure that $b + c = a$ and that the digits rotate so that the first digit becomes the last. This would become $100b + 10c + a$. If the answer is divisible by 9, then it would have be an integer (no remainder).

Question: $\left(\frac{(100a+10b+c)-(100b+10c+a)}{9}\right) = \text{integer}$

Simplify: $\left(\frac{99a-90b-9c}{9}\right)$ Highest common factor = 9 as 99, 90, 9 are all divisible by 9

Factor out the HCF to get: $\left(\frac{9(11a-10b-c)}{9}\right)$ 9 is always divisible by 9, so clearly this will always work

Simplify: $11a - 10b - c$ This becomes important later on.

Now to try some examples:

$$\begin{array}{r} 642 \\ -\ 426 \\ \hline 216 \end{array}$$

Digital root of 216 = 9 $216 \div 9 = 24$

$$11a - 10b - c$$
$$66 - 40 - 2 = 24$$
$$24 \times 9 = 216$$

What is really interesting is that the digital root of the answer is 9, and that the simplified expression calculates how many times the answer is divisible by 9. This is because the simplified expression represents the larger number minus the smaller number, divided by 9.

$$\begin{array}{r} 835 \\ -\ 358 \\ \hline 477 \end{array}$$

Digital root of 477 = 18 = 9 $477 \div 9 = 53$

$$11a - 10b - c$$
$$88 - 30 - 5 = 53$$
$$53 \times 9 = 477$$

$$\begin{array}{r} 909 \\ -\ 099 \\ \hline 810 \end{array}$$

Digital root of 810 = 9 $810 \div 9 = 90$

$$11a - 10b - c$$
$$99 - 0 - 9 = 90$$
$$90 \times 9 = 810$$

The digital root of any number is the sum of all the digits added together. If the result is not a single digit, then repeat the process until this is achieved. E.g. $679421 \rightarrow 29 \rightarrow 11 \rightarrow 2$, digital root of 679421 is 2.

I first read about digital roots about two years ago when working on *nrich.maths.org/33* (via the linked article).

Because a digital root must be a single digit, it can only be 0 to 9. (DR of 0 = 0).

If the digital root of a number is exactly 9, then that number is a multiple of 9.

If any number is not divisible by 9, then its digital root is the remainder left over after being divided by 9.

Because all of my reversals had a digital root of 9, they are all divisible by 9.

Integer	Digital Root	Divisibility by 9
243	9	$243 \div 9 = 27$
242	8	$242 \div 9 = 26\,r8$
241	7	$241 \div 9 = 26\,r7$
240	6	$240 \div 9 = 26\,r6$
239	$14 \rightarrow 5$	$239 \div 9 = 26\,r5$
238	$13 \rightarrow 4$	$238 \div 9 = 26\,r4$
237	$12 \rightarrow 3$	$237 \div 9 = 26\,r3$
236	$11 \rightarrow 2$	$236 \div 9 = 26\,r2$
235	$10 \rightarrow 1$	$235 \div 9 = 26\,r1$
234	9	$234 \div 9 = 26$

There is a pattern to Alison and Charlie's questions, so I can write a general equation:

$$\left(\frac{(10a+b) \pm n}{2}\right) = 10b + a$$
$$10a + b \pm n = 20b + 2a$$
$$8a \pm n = 19b$$

To find the answer I looked for a ratio of $8 \pm n{:}19$, e.g:

$8a + 11 = 19b$ where $a = 1, b = 1$,
$8a + 30 = 19b$ where $a = 1, b = 2$,
$8a + 49 = 19b$ where $a = 1, b = 3$...

E.g. Choose a two digit number add 11 (or 30, or 49 etc), divide this sum by two, and reverse the digits to find your original number. Which number was chosen? $(11 + 11) \div 2 = 11 \rightarrow 11$, $(12 + 30) \div 2 = 21 \rightarrow 12$, $(13 + 49) \div 2 = 31 \rightarrow 13$ These are just three examples, but there are many more I could have chosen.

You can also divide by 4, if you adjust the general equation:

$$\left(\frac{(10a+b) \pm n}{4}\right) = 10b + a$$
$$10a + b \pm n = 40b + 4a$$
$$6a \pm n = 39b$$

This time the ratio is $6 \pm n{:}39$, I chose $6a + 60 = 39b$ where $a = 3, b = 2$, so the question could read: Choose a two digit number add 60, divide this sum by four, and reverse the digits to find your original number. Which number was chosen? Answer: 32 as $(32 + 60) \div 4 = 23$ and 23 reversed is 32. Because *n* is a variable, I could have chosen many values for *a*, *b*, so long as the ratio $6 \pm n{:}39$ works.

Next I multiplied by 9, so that the answer remained divisible by 9, and divided by 6:

Expression: $$9\left(\frac{(10a+b) \pm n}{6}\right) = 10b + a$$

Multiply by LCM... $$18\left(\frac{(10a+b) \pm n}{6} = \frac{10b + a}{9}\right)$$

...to get: $30a + 3b \pm 3n = 20b + 2a$

Simplify: $28 \pm 3n = 17b$

e.g. $a = 1, n = 2, b = 2$

The ratio is $28 \pm 3n{:}17$, so I compared multiples of 28 and multiples of 17, looking for a difference that was divisible by 3. $(1 \times 28) + (2 \times 3) = (2 \times 17)$ is one example

Substitute: $$9\left(\frac{(12+2)}{6}\right) = 21 \quad \text{as} \quad \frac{9 \times 14}{6} = \frac{126}{6} = 21 \quad \text{and 21 reversed is 12}$$

The question could read: chose a two digit number add 2, multiply the sum by 9, divide the answer by 6, and reverse the digits to get your original number. Which number was chosen? Answer: 12. These are just some examples, but you could come up with many more. It's just about finding the correct ratio and adjusting *n*.