3-times table

Difference of Two squares

E.g. 6

$7^2 - 5^2 = 49 - 25$
$= 24$

$\frac{24}{3} = 8$

E.g. 12

$13^2 - 11^2 = 169 - 121$
$= 48$

$\frac{48}{3} = 16$

This demonstrates that the difference of two squares of two numbers one greater and one less than a multiple of three is a multiple of three.

Proof

Let $3n$ be a multiple of 3, $n \in \mathbb{N}, n > 1$

$\therefore$ Difference of squares of numbers on either side =

$(3n+1)^2 - (3n-1)^2$

$= 9n^2 + 6n + 1 - (9n^2 - 6n + 1)$

$= 12n$

$\frac{12n}{3} = 4n$ $\therefore$ $12n$ is a multiple of 3, as 3 divides it exactly.

5-times table

Eg 10

$11^2 - 9^2 = 121 - 81$
$= 40$

$\frac{40}{5} = 8$

E.g. 15

$16^2 - 14^2 = 256 - 196$
$= 60$

$\frac{60}{5} = 12$

This demonstrates that a similar pattern exists for the 5-times table, which can be proved in a similar way.

Proof

Let $5n$ be a multiple of 5, $n \in \mathbb{N}, n > 1$

$\therefore$ Difference of squares of numbers on either side =

$(5n+1)^2 - (5n-1)^2$

$= 25n^2 + 10n + 1 - (25n^2 - 10n + 1)$

$= 20n$

$\frac{20n}{5} = 4n$ $\therefore$ $20n$ is a multiple of 5 as 5 divides it exactly.

Now, in order to ascertain whether a similar relationship exists for other times tables, one must define a multiple of α where $\alpha \in \mathbb{N}$, $n \in \mathbb{N}, n > 1$

Let αn be a multiple of α. (This is the same step as before, just with the α times table, rather than the 3 or 5 times tables).

$\therefore$ Difference of squares of numbers on either side =

$(\alpha n + 1)^2 - (\alpha n - 1)^2$

$= \alpha^2 n^2 + 2\alpha n + 1 - (\alpha^2 n^2 - 2\alpha n + 1)$

$= 4\alpha n$

Dividing by α, which is the times table:

$\frac{4\alpha n}{\alpha} = 4n$ $\therefore$ $4\alpha n$ is a multiple of α, as α divides it exactly.

This proves that a similar relationship exists for every times table, as α can be any natural number.

Extension

Using the 3-times table and e.g 6

the difference of squares of the numbers 2 above and 2 below are:

$8^2 - 4^2 = 64 - 16$
$= 48$

$\frac{48}{3} = 16 \therefore$ 48 is a multiple of 3, as 3 divides it exactly.

Using the 5-times table e.g 20

$22^2 - 18^2 = 484 - 324$
$= 160$

$\frac{160}{5} = 32 \therefore$ 160 is a multiple of 5, as 5 divides it exactly.

Now, one can prove algebraically that the same is true for all multiples of α, $\alpha \in \mathbb{N}$, but one can ~~go~~ do more. Instead, it shall be proved that the same is true for all multiples of α, $\alpha \in \mathbb{N}$ and all pairs β above and below α, $\beta \in \mathbb{N}$. (This means ~~that~~ for all numbers 2 above and below, 3 above and below, etc.)

Let αn once more be a multiple of α, $\alpha \in \mathbb{N}$.

Instead of using integers to describe how far above and below ~~the~~ αn one is going, a variable β shall be introduced, $\beta \in \mathbb{N}$ (β can be any natural number (0 is not natural)). $n \in \mathbb{N}, n > 1$ (~~as~~ this ~~need~~ does not work for $n = 1$).

$\therefore$ The difference of squares of the numbers β above and below $\alpha n =$

$$(\alpha n + \beta)^2 - (\alpha n - \beta)^2$$

$$= \alpha^2 n^2 + 2\alpha\beta n + \beta^2 - (\alpha^2 n^2 - 2\alpha\beta n + \beta^2)$$

$$= 4\alpha\beta n$$

Dividing by α, which is the times-table (This is the same as dividing by 3 or 5 earlier)

$= \frac{4\alpha\beta n}{\alpha} = 4\beta n \therefore$ $4\alpha\beta n$ is a multiple of α as α divides it exactly.

So, it has been proved that the difference of squares of ~~all~~ ~~integers~~ 2 natural numbers greater than and less than a fixed natural number by the same amount is a multiple of that fixed natural number.

E.g for large numbers

683, using the 683rd times table, 50 above and below.

$(683 + 50)^2 - (683 - 50)^2$
$= 733^2 - 633^2$
$= 537289 - 400689$
$= 136600$

$\frac{136600}{683} = 200 \therefore$ 136600 is a multiple of 683, as 683 divides it exactly.