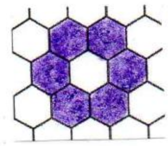
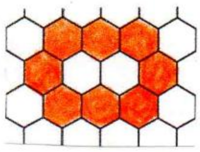
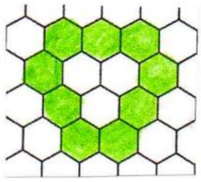
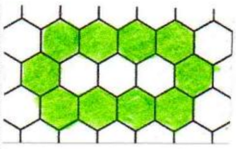
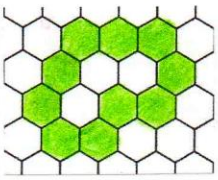
Regular Hexagon Loops: Stage 3**

Zach Thompson

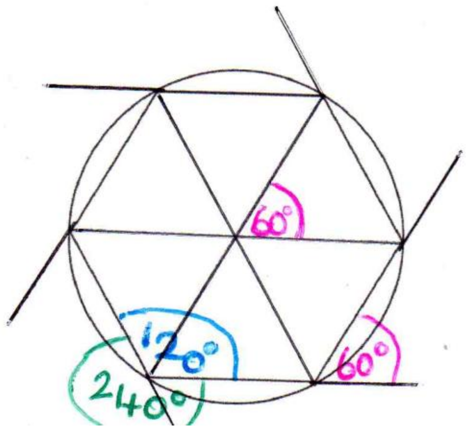
I started by looking at the different ways that you can arrange a closed chain of hexagons, to grow a "hole" in the middle of a loop.

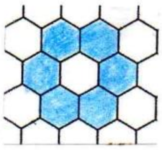
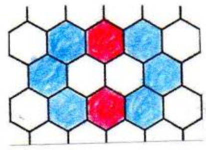
Number of "Blanks" (central hole)	Number of Tiles in Loop	Inside Perimeter of Loop (nr of sides)	Outside Perimeter of Loop (nr of sides)
 1	6	6	18
 2	8	10	22
 3	9	12	24
 3	10	14	26
 3	10	14	26

Although each hexagon has the same area, how you arrange the blank hexagons determines the perimeter.

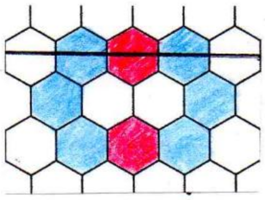
Note that the external perimeter of the central "hole" is the internal perimeter of the loop chain. If 3 blank tiles are clustered about a central point, then each tile loses (or shares) 2 edges, leaving a perimeter of (3 x 4) = 12 sides, but if 3 blank tiles are joined in either a straight line, or if the third tile is at a 120° angle, then only the central tile shares 2 edges, generating a perimeter of (2 x 5) + (1 x 4) = 14 sides.

A regular hexagon is made up of 6 equilateral triangles:

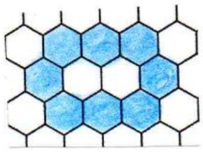
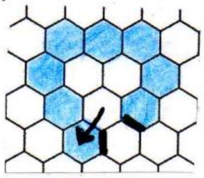
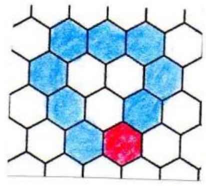


This is the smallest loop possible:		To grow the loop, you have to add 2 tiles:
	→	
6 loop tiles, 6 inside perimeter 18 outside perimeter		6 + 2 = 8 loop tiles, 6 + 4 = 10 inside perimeter 18 + 4 = 22 outside perimeter

The 2 additional tiles are inserted at a 180° angle to the existing loop and add 4 faces to both the inside and outside perimeter:

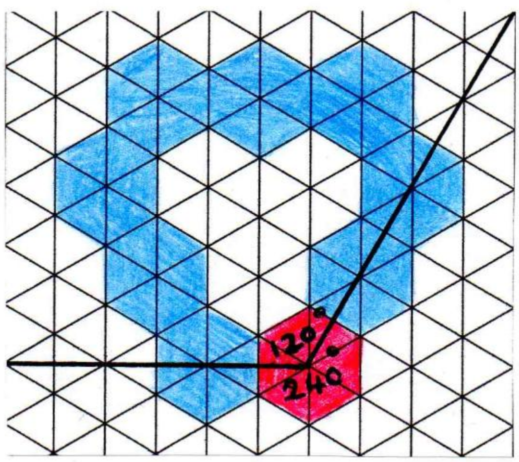


To grow a loop from 2 holes to 3, you have to rearrange the existing hexagons before adding a new tile:

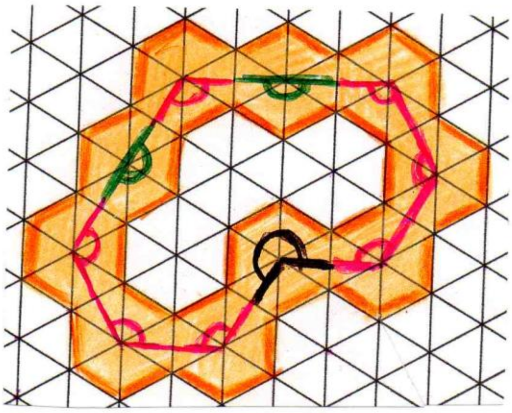
Existing loop:		Rearrange, before adding new tile:		Now add a new tile:
	→		→	
8 loop tiles, 10 inside perimeter 22 outside perimeter		8 loop tiles, 11 inside perimeter 21 outside perimeter		8 + 1 = 9 loop tiles, 11 + 1 = 12 inside perimeter 21 + 3 = 24 outside perimeter

Excluding the 2 faces marked in bold (which will disappear when the new tile is introduced), the inside perimeter increases to 11, but the outside perimeter shrinks to 21.

When the new tile is added, it sits at an angle of 120° to the inside perimeter, and adds 1 face to this. Logically, this means that the new tile sits at an angle of 240° to the outside perimeter, and so contributes 3 faces. This implies that angles influence perimeter. Triangular paper makes this clearer.

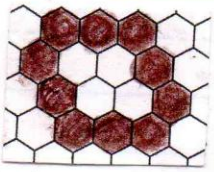
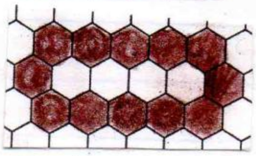
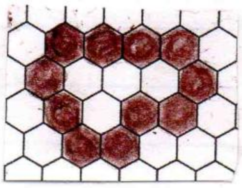
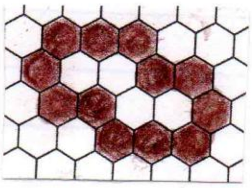
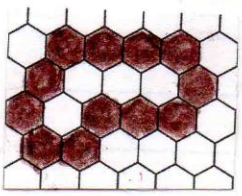
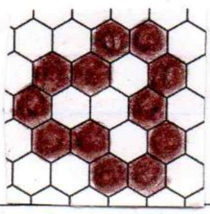


This loop illustrates how this happens: 120° angles are marked in red, 180° angles are marked in green, and 240° angles are marked in black:



It's clear that when a hexagon is added to a chain at a 120° angle it contributes 1 side, a 180° angle adds 2 sides and a 240° angle adds 3 sides to any given perimeter.

To understand how the number of tiles influences perimeter, I gathered some more data:

Number of "Blanks" (central hole)	Number of Tiles in Loop	Inside Perimeter of Loop (nr of sides)	Outside Perimeter of Loop (nr of sides)
 4	10	14	26
 4	12	18	30
 4	11	16	28
 4	12	18	30
 4	12	18	30
 4	12	18	30

Next I compared the data and very quickly found that the rule is (2n – 6) for the inside perimeter, and (2n + 6) for the outside perimeter, where n is the number of loop tiles. The shaded areas are workings.

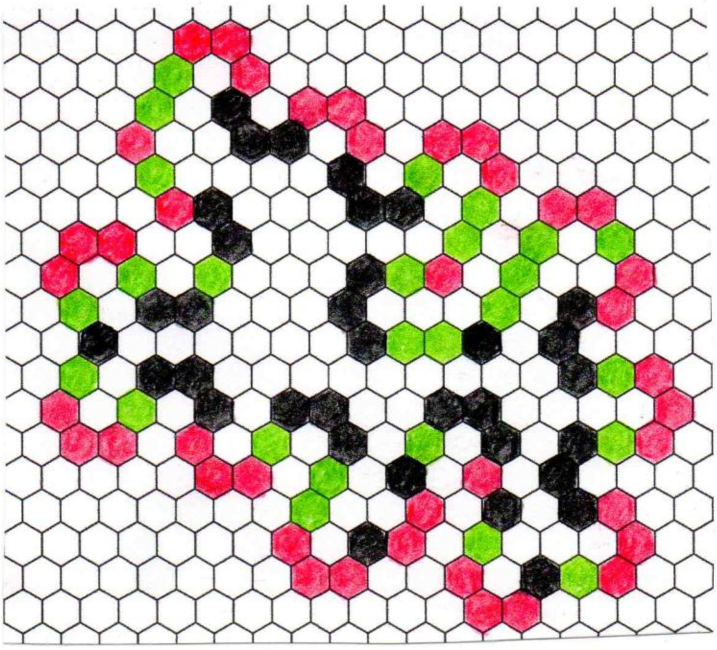
Number of Loop Tiles (n)	Rule for Inside Perimeter 2n – 6	Number of Inside Perimeter sides
6	12 – 6 = 6	6
8	16 – 6 = 10	10
9	18 – 6 = 12	12
10	20 – 6 = 14	14
11	22 – 6 = 16	16
12	24 – 6 = 18	18

Number of Loop Tiles (n)	Rule for Outside Perimeter 2n + 6	Number of Outside Perimeter Sides
6	12 + 6 = 18	18
8	16 + 6 = 22	22
9	18 + 6 = 24	24
10	20 + 6 = 26	26
11	22 + 6 = 28	28
12	24 + 6 = 30	30

I tried to link some ideas, by looking an irregular loop. I coloured the tiles to make it easier to count.

Because a regular hexagon loop is a polygon, every tile sits between two others within the chain. Angles are calculated by imagining a line through the centre of three adjacent tiles, to find the angle at which the central tile joins the loop.

Red tiles = 120° angle to the inside perimeter, and 240° angle to the outside perimeter,
Green tiles = 180° angle to both the inside and outside perimeter,
Black tiles = 240° angle to the inside perimeter, and 120° angle to the outside perimeter.



There are 100 tiles in total:

Inside Perimeter

Red = 40 x 1 = 40 sides
Green = 26 x 2 = 52 sides
Black = 34 x 3 = 102 sides
Total number of sides = 194 sides

2n – 6, n = 100, leading to: 200 – 6 = 194 sides

Outside Perimeter

Red = 40 x 3 = 120 sides
Green = 26 x 2 = 52 sides
Black = 34 x 1 = 34 sides
Total number of sides = 206 sides

2n + 6, n = 100, leading to: 200 + 6 = 206 sides

The rule seems to hold.

I then counted the internal angles and noticed that they are all either 120° or 240°

The sum of the interior angles of a n–gon is (n – 2)180   n = 194   192 x 180 = 34,560°

194 sides = 194 angles

Red tiles sit at 120° to their neighbours and contribute 1 side to the internal perimeter.
Each red tile contributes 1 x 120° to the sum of interior angles.

Green tiles sit at 180° to their neighbours and contribute 2 sides to the internal perimeter.
Each green tile contributes (120° + 240° = 360°) or 2 x 180° to the sum of interior angles

Black tiles sit at 240° to their neighbours and contribute 3 sides to the internal perimeter.
However, each black tile contributes (120° + 240° + 240° = 600°) or 3 x 200° to the sum of interior angles

We can now calculate:

Red = 40 x 1 x 120° = 4,800°
Green = 26 x 2 x 180° = 9,360°
Black = 34 x 3 x 200° = 20,400°
Sum of interior angles = 34,560°

I did go on and draw much more complex loops, the same patterns held.