Puzzling Place Value: Stage 4** Zach Thompson

$$ABC = \text{any 3 digit number} = 100A + 10B + C$$

This is because of place value where z in the hundreds column is worth 100 times its unit value and z in the tens column is worth 10 times its unit value: e.g. $A = 5$, $B = 2$, $C = 6$ so $(100\times5) + (10\times2) + (1\times6) = 526$

AB = any two digit number $= 10A + B$, e.g. $A = 8$, $B = 3$ $(10\times8) + (1\times3) = 83$. If you want to reverse the digits, $10A + B$ becomes $10B + A$ ... $(83 \rightarrow 38)$. The question also says that I must use the digits from 1 to 9. I can now use all of this to solve the puzzles by rewriting the problems using algebra.

1) $4(A + B) = 10A + B$

$4A + 4B = 10A + B$

$3B = 6A$

$B = 2A$ Because I can only use the digits 1 to 9, $A < 5$ so the only possible values for A are 1, 2, 3 or 4 $A = 5, B = 10$, is not possible as B has to be a single digit

These are the only four solutions: 12, 24, 36 and 48

Variables	$10A + B$	$4(A + B)$
$A = 1, B = 2$	$10+2 = 12$	$12 = 4(1+2)$
$A = 2, B = 4$	$20+4 = 24$	$24 = 4(2+4)$
$A = 3, B = 6$	$30+6 = 36$	$36 = 4(3+6)$
$A = 4, B = 8$	$40+8 = 48$	$48 = 4(4+8)$

2) $(100A + 10B + C) + (10A + B) + C = 300$ becomes $110A + 11B + 2C = 300$

I noticed a common factor in the multiples of A, B, so factored this out. $11(10A + B) = 300 - 2C$

I substituted $C = 9$, because it's the largest digit to see what would happen: $11(10A + B) = 300 - 18$

$$11(10A + B) = 282$$

$$\text{But } \frac{282}{11} \neq \text{integer}$$

This meant that I was looking for an even 2 digit number between 282 and 298 which is divisible by 11. The number has to be even as C must be an integer in the range 1 to 9, so $2C$ and therefore $300 - 2C$ are both even.
The maximum and minimum values for $2C$ are: $C = 9$, $300 - 2(9) = 282$
$C = 1$, $300 - 2(1) = 298$.

There is only one number divisible by 11 that is in this range: 286, and $286 \div 11 = 26$, so $A = 2, B = 6, C = 7$

$$11(10A + B) = 300 - 2C$$

$$11(26) = 300 - 14$$

$$286 = 286$$

So the sum should read: $267+26+7 = 300$

3)
$$\begin{array}{r} 10A + B \\ -(A + B) \\ \hline 9A \end{array} \quad + \quad \begin{array}{r} 10B + A \\ -(B + A) \\ \hline 9B \end{array} \quad = 9(A + B)$$

You will always get a multiple of 9 because 9 is the Highest Common Factor in $9A + 9B$. Any 2 digit positive number will give you this result if you follow the steps in the question, e.g.

$$\begin{array}{r} 82 \\ -\ 10 \\ \hline 72 \end{array} \quad + \quad \begin{array}{r} 28 \\ -\ 10 \\ \hline 18 \end{array} \qquad \begin{array}{l} A = 8, B = 2 \\ 9A = 9\text{x}8 \text{ and } 9B = 9\text{x}2 \\ = 90 \text{ or } 9(8+2) \end{array}$$

The digital root of 72, 18, 90 is exactly 9, so these numbers are all divisible by 9.
I explore digital roots in my extension.

It doesn't matter which two digits you use, you will always get a multiple of 9, because of the algebra.

4)

$$\begin{array}{rl} & 10A+B \\ & 10A+C \\ & 10B+A \\ & 10B+C \\ & 10C+A \\ + & 10C+B \\ \hline & 22A+22B+22C \end{array}$$

Three digits are A, B, C. Six possibilities sum to $22A+22B+22C$. Factor out the HCF of 22 to get: $22(A+B+C)$, then divide by the sum of the three digits $(A+B+C)$:

$$\frac{22(A+B+C)}{(A+B+C)}$$

The sum of digits $(A+B+C)$ will cancel out, leaving only a number that is a multiple of 22.

5)

$$\begin{array}{c} 10A+B \\ -(10B+A) \\ \hline (9A-9B) \end{array} + \begin{array}{c} 10B+C \\ -(10C+B) \\ \hline (9B-9C) \end{array} + \begin{array}{c} 10A+C \\ -(10C+A) \\ \hline (9A-9C) \end{array} = 18A-18C, \text{ or } 18(A-C) \qquad A>B>C$$

What is also noticeable is that the first two columns sum to make the third:

$$(9A-9B) + (9B-9C) = (9A-9C)$$ So the expression is also $2(9(A-C))$

So the answer is always a multiple of 18.

6)

$$100A+10B+C=(10A+B)+(10B+C)+(10C+A)$$

$$100A+10B+C=11A+11B+11C$$ So we are looking for a multiple of 11

$$89A-B-10C=0$$

$$89A=10C+B$$

Because we can only use the digits 1 to 9, the only possible solution for A is $A = 1$, because $A = 2$ gives a three digit number. This leaves $C = 8$, $B = 9$: $198 = 19 + 98 + 81$, and $198 \div 11 = 18$

7)

$$(10A+B)+(10C+D)=(10D+C)+(10B+A)$$

$$9A+9C=9B+9D$$

$$A+C=B+D$$ is the general rule

We can only use the digits 1 to 9, so it's just about finding combinations that work:

A	+	C	=	B	+	D	
1	+	4	=	2	+	3	12+43=34+21
1	+	4	=	3	+	2	13+42=24+31
4	+	1	=	2	+	3	42+13=31+24
4	+	1	=	3	+	2	43+12=21+34 The combination {1,4,2,3} has 4 possibilities
1	+	5	=	2	+	4	12+54 = 45+21
1	+	5	=	4	+	2	14+52 = 25+41
5	+	1	=	2	+	4	52+14 = 41+25
5	+	1	=	4	+	2	54+12 = 21+45, again 4 possibilities

I found 34 different combinations (there are many repeats) each with 4 possibilities, so I think that there may be 136 examples that fit the rule.

The reason why this works is that the ratios remain the same 12+43=34+21, as 12+9 = 21 while 43 – 9 = 34

Extension:

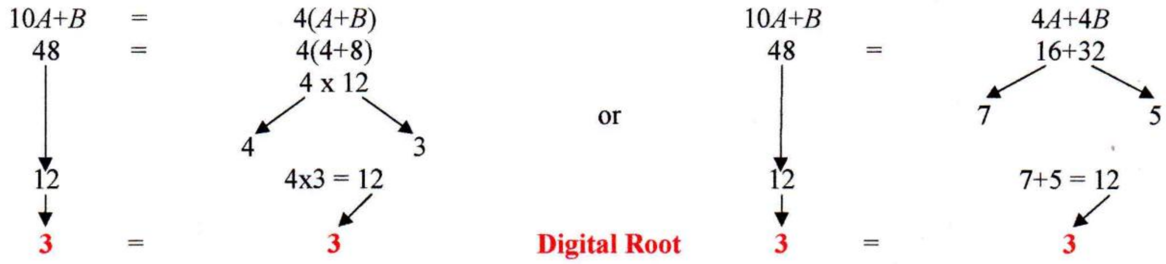
The questions focused on algebra, but I noticed a strong pattern in the digital roots (DR).

The digital root of any number is the sum of all the digits added together. If the result is not a single digit, then repeat the process until this is achieved. E.g. 84925137 → 39 → 12 → 3, digital root of 84925137 is 3.

A digital root can only be a single digit. If a number is exactly divisible by 9, its digital root is also 9. If any number is not divisible by 9, then its digital root is the remainder left over after being divided by 9. The digital root of 0 is 0 (and nothing else).

Although it seems really obvious, if you add numbers, the sum will also be the sum of their digital roots. I first realised this when gazing at Fibonacci numbers and working out their digital roots, which have a cyclical pattern of 24 digits which always repeats.

Q1) $B = 2A$, giving an answer of 12, 24, 36, 48 which have a DR pattern of 3, 6, 9, 3. This is the DR pattern of the three times table, and it's the 1:2 ratio which means that the numbers are divisible by 2,3,4,6 and 12. The arrows show the path to calculate the DR.



When you multiply a number its DR is multiplied too, e.g:

$$\begin{tikzcd}
12 \arrow[d] & \times & 2 \arrow[d] & = & 24 \arrow[d] & \\
3 & \times & 2 & = & 6 & \text{Digital root}
\end{tikzcd}$$

So I tried some bigger numbers:

$$\begin{tikzcd}
5104 \arrow[d] & \times & 36 \arrow[dd] & = & 183744 \arrow[d] & \\
10 \arrow[d] & & & & 27 \arrow[d] & \\
1 & \times & 9 & = & 9 & \text{Digital Root}
\end{tikzcd}$$

Which made me think that it would probably work with division:

$$\begin{tikzcd}
183744 \arrow[d] & \div & 36 \arrow[dd] & = & 5104 \arrow[d] & \\
27 \arrow[d] & & & & 10 \arrow[d] & \\
9 & \div & 9 & = & 1 & \text{Digital Root}
\end{tikzcd}$$

But when I tried a few other examples, I realised that it wasn't always so simple:

$$\begin{tikzcd}
7065913 \arrow[d] & \times & 395 \arrow[d] & = & 2791035635 \arrow[d] & \\
31 \arrow[dr] & & 17 \arrow[dl] & & 41 \arrow[dd] & \\
& 4 \times 8 = 32 \arrow[d] & & & & \\
& 5 & & = & 5 & \text{Digital Root}
\end{tikzcd}$$

This was tricky:

$$\begin{tikzcd}
2791035635 \arrow[d] & \div & 395 \arrow[d] & = & 7065913 \arrow[d] \\
41 \arrow[dr] & & 17 \arrow[dl] & & 31 \arrow[ddd] \\
& 5 \div 8 = 0.\underline{625} \arrow[d] \arrow[ddl, bend right] \arrow[dddr, bend right] & & & \\
& 13 \arrow[d] & & & \\
& 4 & & = & 4 \\
3+2 \arrow[d] & & & & \\
32 & \div & 8 & = & 4
\end{tikzcd}$$

Either:

calculate DR of 625 (not possible for recurring decimals), or

go back and rephrase DR of 5 as 32 (3+2 = 5)

Q2) was straightforward:

$$\begin{tikzcd}
267 \arrow[d] & + & 26 \arrow[dd] & + & 7 \arrow[ddl] & = & 300 \\
15 \arrow[dr] & & & & & & \\
& 6 & + 8 & + 7 & = 21 \arrow[d] & & \\
& & & & 3 & = & 3 & \text{Digital root}
\end{tikzcd}$$

Q3) It was difficult to reason out the patterns in the digital roots, so I had to think really hard.
Using the values $A = 3$, $B = 6$

$$\begin{array}{r} 10A + B \\ -(A+B) \\ \hline 9A \end{array} + \begin{array}{r} 10B + A \\ -(B+A) \\ \hline 9B \end{array} = 9(A+B)$$

$$\begin{tikzcd}
\text{Number} & & \text{DR} & \text{Number} & & \text{DR} \\
36 & & 9 & 63 & & 9 \\
\underline{-\ 9} & & \underline{-9} & \underline{-\ 9} & & \underline{-9} \\
27 \arrow[dr, bend right] & \neq & 0 & 54 \arrow[dr, bend right] & \neq & 0 \\
& =9 \arrow[dr, bend right] & & & =9 \arrow[dl, bend left] & \\
& & 9+9=18 \arrow[d] & & & \\
& & 9 & \text{Digital Root} & &
\end{tikzcd}$$

This proves that you always get a multiple of 9, as your answer will always have a digital root of 9. But it also shows that when you subtract DR's, you can get a non-positive answer.

This comes back to the fact that digital roots are multiples of 9, plus a remainder, it's just that when the number is a multiple of 9, the remainder is zero.

If I used this model I would always end up with a multiple of 9, so I subtracted a different set of numbers:

$$\begin{tikzcd}
91 \arrow[d] & - & 35 \arrow[dd] & = & 56 \arrow[d] & \\
10 \arrow[d] & & & & 11 \arrow[ddd] & \\
1 \arrow[dr] & - & 8 \arrow[dl] & & & \\
 & 1-8=\underline{-7} \arrow[d] & & & & \\
 & \text{and } 9-7 & & = & 2 & \text{Digital Root}
\end{tikzcd}$$

Every number is made up of multiples of 9, plus a remainder, so 91 – 35 = 56 can be written as:

$$\begin{array}{r} (10 \text{ x } 9) + 1 \\ -\,(3 \text{ x } 9) + 8 \\ \hline (7 \text{ x } 9) - 7 \end{array} = 56$$

But (7 x 9) – 7, can also be rewritten as (6 x 9) + (1 x 9 – 7), or (6 x 9) + 2 = 56
It's a bit like carrying back with multiples of 9, where 56 is divisible by 9, with a remainder of 2.

Q4) e.g. $A = 5$, $B = 6$, $C = 7$ becomes 56+57+65+67+75+76 = 396, (5+6+7 = 18) and 396÷18 = 22
↘ ↘ ↘ ↘ ↘ ↘ ↘ ↓ ↙ ↙
The digital root equivalent is: 2 + 3 + 2 + 4 + 3 + 4 (= 18) = 9 9÷9 ≠ 4

The addition bit is easy, but the division bit doesn't work unless you go back and rephrase a DR of 9 as 36 (3+6 = 9), so 36÷9=4. In this case the digits summed to 9, so I used another example:

e.g. $A = 8$, $B = 4$, $C = 3$ becomes 84+83+48+43+38+34 = 330, (8+4+3 = 15) and 330÷15 = 22
↘ ↘ ↘ ↘ ↘ ↘ ↘ ↓ ↙ ↙
The digital root equivalent is: 3 + 2 + 3 + 7 + 2 + 7 (= 24) = 6 6÷6 ≠ 4,

This proves that 330 ÷ 9 = 36, remainder 6, but it doesn't show that the final answer is a multiple of 22 which shows the limitation of digital roots, **algebra finds the rule, DRs show patterns in number**. What it does show is that the DR of the numerator and denominator are the same. If you go back and rephrase a DR of 6 as 24 (2+4 = 6), then 24÷6 = 4, but this is just my way of imagining how this can work.

Q5) e.g. $A = 7$, $B = 2$, $C = 1$, the digital root equivalent sum is below:

(72 – 27 = 45) + (21 – 12 = 9) + (71 – 17 = 54) and 45+9+54 = 108 108÷18 = 6
↓ ↙ ↙ ↓ ↓ ↙ ↓ ↓ ↙
(9 – 9 = 9) + (3 – 3 = 9) + (8 – 8 = 9) Each answer is divisible by 9, so the DR is 9, with 0 remainder, so the subtraction bit works, because I carried back a multiple of 9 to get a positive DR. The division is tough: 108÷18 = 6 becomes 9÷9 ≠ 6, but if you go back and rephrase a DR of 9 as 54 (5+4 = 9), then 54÷9 = 6.

Q6) was straightforward again 19 + 98 + 81 = 198, DR equivalent sum 1 + 8 + 9 (= 18) = 9

Q7) There are lots of different possibilities, all are straightforward additions, so the equivalent DR sums will also add up. E.g: $A = 6$, $C = 9$, $B = 7$, $D = 8$, becomes 67+98 = 89+76, DR equivalent sum 3=3, but the general rule $A + C = B + D$, gives 6+9 = 7+8, DR equivalent 6 = 6, which is interesting. Similar patterns exist for the other examples also.

This really shows how beautiful algebra is, how easily it finds a relationship. Digital roots are also powerful, but limited. Doing this helped me understand how digital roots subtract and to imagine a simple way of showing how they may divide.