In this project we would like to explain how we solved the mathematical problem posted in http://nrich.maths.org/11301.

	y-axis is an asymptote			Passes through the origin
$x = 1$ is a root		$y = \lvert x-1 \rvert$	$y = -3x + 3$	
Has exactly two roots		$y = x(x-2)$		
	$y = \frac{1}{x}$ $x \neq 0$	$y = \frac{1}{(x-1)^2}$ $x \neq 1$	$y = \frac{3}{x+1}$ $x \neq -1$	
$y \to \infty$ as $x \to \infty$			$y = 2 + (x-1)^4$	

This was the original table, with which we started. We thought that the best way of solving the task was to, in first place, decide what the headings were. The three headings written in bold at the end are the ones we thought best fitted the functions.

The first heading we filled in had functions like $y = \frac{1}{x}$ or $y = \frac{1}{(x-1)^2}$

All these function have something in common, as x tends to infinity, the functions tend to 0, which means that they have an asymptote at y=0, or that the ***X-axis is an asymptote.***

With this horizontal heading we had completed all of the first vertical column, which were the headings of the four rows in our table. Then we decided to start with the two headings missing from the first horizontal row. The first one had functions like $y = \frac{1}{(x-1)^2}$ $y = |x-1|$ or .

These are just transformations of other simpler functions like $|x|$ and $\frac{1}{x^2}$. These two functions are vertically symmetric, as $f(-x) = f(x)$, so its transformations are also symmetric. ***Symmetrical at x = 1.***

The last heading had functions like $y = -3x + 3$ or $y = 2 + (x-1)^4$. We decided that, to solve this problem, we could input values of *x* like 0 or infinity, to look at any patterns that could appear. When we tried with *x*=0, we realised that the y was always three. This meant that the heading was ***Intercepts the y-axis at 3.***

Once we concluded with the headings we decided to move onto finding suitable functions that fitted with the characteristics (headings) of the table.

The first function we added had to have $x = 1$ as a root and have an asymptote at the y-axis. The function that we thought of was $y = ln(x)$ This is just another expression in disguise: $x = e^y$. When *y*=0, (ie the root), x = e^0 = 1. The y-axis (or the line x = 0) is an asymptote because as y tends to negative infinity, it becomes $x = \frac{1}{e^y}$ so as *y* tends to infinity, x approaches 0.

The second function we looked at had to have $x = 1$ as a root and needed to pass through the origin. We thought that the function $y = x^2(x-1)$ had both characteristics. For *x* to be a root, *y* had to be equal to zero. This function was ideal as the roots were 1 and 0. For the line to pass through the origin when *x* was equal to zero, *y* had also to be 0. This was easy as we had already addressed this problem during the procedure of the last problem.

Thirdly, we had to find a function with two roots and the y-axis as an asymptote. The function

$y = \frac{1}{x^2} - 3$ for $x \neq 0$ had all the necessary properties. To find the roots, we can set y to be equal to 0 and rearrange:

$$0 = \frac{1}{x^2} - 3$$

$$3 = \frac{1}{x^2}$$

$$x^2 = \frac{1}{3}$$

$x = \pm\sqrt{\frac{1}{3}}$. So it has two roots.

At x=0, the function does not exist, as

$\lim_{x\to 0} \frac{1}{x^2} - 3 = \frac{1}{0}$, which is indeterminate. So we need to do the limits from the left and the right:

$\lim_{x\to 0^-} \frac{1}{x^2} - 3 = +\infty$ and

$\lim_{x\to 0^+} \frac{1}{x^2} - 3 = +\infty$ Notice that when the function approaches 0 from both directions it tends to infinity, as the x^2 gets rid of the sign. Which means that the limit ($+\infty$) exists.

In fourth place, we had to to look for a function with two characteristics. The first one was that it had to intercept the *y-axis* at 3, and the second property was that it had to have exactly two roots. This meant that our function had to be quadratic, $y = ax^2 + bx + c$, and one of its roots had to be 3. We decided to factorise the equation to $y = (x+a)(x-3)$, having one of the roots already being equal to 3. Then *a* could be

any real number, so we just decided to make it easy by having another root equal 1. We expanded the brackets, an unnecessary step really, to get the final function
$y = x^2 - 2x - 3$

The fifth function had to have exactly two roots, one of which had to be the origin. A function that works is $y = x(x-5)^2$ To find the roots, we can set y=0 and solve. It can be solved easily if we look at what values of x make the whole thing 0. The first value is x=0 and the second value is x=5. So we have two roots, one which is at (0,0) or the origin. The square at the end of (x - 5) doesn't affect the number of roots, the function just behaves differently at the root.

The sixth function we worked out had to pass through the origin and have the x-axis as an asymptote. Some asymptotes can be crossed, namely horizontal and oblique asymptotes, but not vertical asymptotes. The function we came up with is

$$y = \frac{x}{(x-1)^2}$$

The origin has the coordinates (0,0), so if it passes through the origin then when *x*=0, *y*=0 as well. If we put 0 into the function, $y = \frac{0}{(0-1)^2}$ which makes y=0 as the numerator is 0.

To show that it has the x-axis as an asymptote, we can see what the function does as $x \to \infty$.
$\lim_{x\to\infty} \frac{x}{(x-1)^2}$. When x is a very large number, both the denominator and the numerator will be large numbers. However, the denominator will be larger, so *y* will approach 0 (ie. the x-axis will be an asymptote).

The seventh function we had to solve had to have the following properties: *the y-axis as an asymptote* and as $x \to \infty, y \to \infty$. The function we chose was $y = \frac{(x^2+1)}{x}$

When x tends to infinity, so does y because the infinity at the numerator is larger than the one at the denominator.
$$\lim_{x\to+\infty} \frac{(x^2+1)}{x} = +\infty$$
At the point x=0, the function does not exist, so if you take the limits as it approaches 0, you can determine if it is an asymptote or not.
$$\lim_{x\to 0^-} \frac{(x^2+1)}{x} = -\infty$$
$$\lim_{x\to 0^+} \frac{(x^2+1)}{x} = +\infty$$
This demonstrates that the y-axis acts as an asymptote.

The final function had to have two properties, it had to pass through the origin and as $x \to \infty$, $y \to \infty$. This one was quite simple as in each of the characteristic the *x* and the *y* had to be the same, in both cases zero and infinity. The function could be $y = x$ or $y = x^2$ or $y = \frac{x^3}{\sqrt{x}}$ …. At the end we just chose the easy option… $y = x$.

At the end the table was the following:

	y-axis is an asymptote	**Symmetrical at $x = 1$**	**Intercepts the *y*-axis at 3**	Passes through the origin
$x = 1$ is a root	$y = ln(x)$	$y = \lvert x - 1 \rvert$	$y = -3x + 3$	$y = x^2(x - 1)$
Has exactly two roots	$y = \frac{1}{x^2} - 3$ $x \neq 0$	$y = x(x - 2)$	$y = x^2 + 2x + 3$	$y = x(x - 5)^2$
***x*-axis is an asymptote**	$y = \frac{1}{x}$ $x \neq 0$	$y = \frac{1}{(x - 1)^2}$ $x \neq 1$	$y = \frac{3}{x + 1}$ $x \neq -1$	$y = \frac{x}{(x - 1)^2}$ $x \neq 1$
$y \to \infty$ as $x \to \infty$	$y = \frac{(x^2 + 1)}{x}$ $x \neq 0$	$y = (x - 1)^2$	$y = 2 + (x - 1)^4$	$y = x$