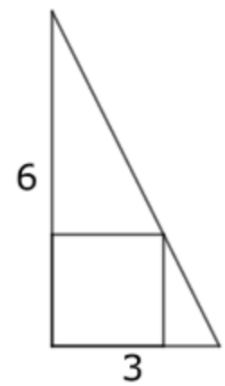
NRICH: The Square Under The Hypotenuse Problem

Abstract

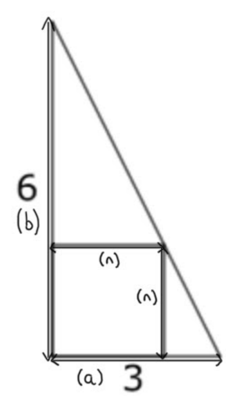
In the diagram of the given mathematical problem on the right hand side, it can be inferred that the right-angled triangle has a base width of 3 units and a height of 6 units (whereby the units are not specified, nor of significant importance). From this problem, the question arises as to how one might go about constructing the square, such that it just touches the hypotenuse[1] of the aforementioned triangle at a particular given point? Hence, using the said polygon, another query that may arise would be as to whether it is possible to calculate the side lengths of the inscribed square? Furthermore, it can also be questioned as to what the positive numerical value for the given square's length would be in the scenario whereby the side lengths of the triangle were 4 units in base length and 12 units in height respectively as opposed to the given numerical values. On a similar note, as expressed in algebraic terms, what would the side length of the square be if the triangle possessed dimensions of 'a' units in base length and 'b' units in height respectively? This research paper will be exploring three methodologies used to solve the plethora of questions in the aforesaid problem and attempt to ascertain as to why these systems of equations prove to be valuable for students and teachers alike.

[1] The hypotenuse refers to the side (that too, the one longest in nature) situated directly opposite to the right angle (an angle with numerical value of 90°) of any given right-angled triangle, regardless of whether it satisfies Pythagoras' Theorem.

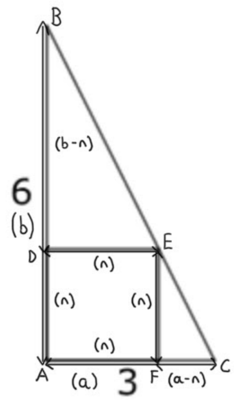
Solutions



Using the diagram above (for the initial solution amidst the total of three devised), let the side length represented with numerical (positive integer) value of 3 units be 'a' for the provided right-angled triangle. Likewise, let the side length with numerical (positive integer) value of 6 units be 'b' for the said triangle. Furthermore, it shall also be taken that the side length of the square inside the said right-angled triangle be 'n' and drawn as shown above. Despite the fact that the use of three variables leads to multivariate expressions and equations it will be seen in due time that this system of equations leads to the desired result in a much swifter manner. Thus, it is evidently seen that the given polygon can be labelled as follows:

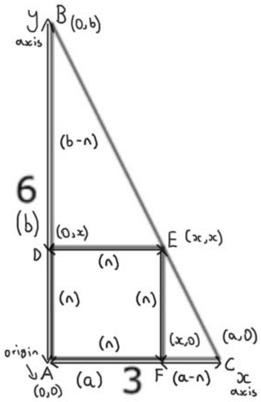


From the labelled polygon above, another fact that can be deduced (and visually represented, for that matter) would be that the sides which are smaller in nature in the given diagram are of similar triangles inside the given triangle. Consequently, it can be inferred that the width and height (respectively, as far it is concerned) of the smaller (and similar, on that note) triangles are '$a - n$' and '$b - n$'. Hence, the previously seen triangle can be further labelled as follows:



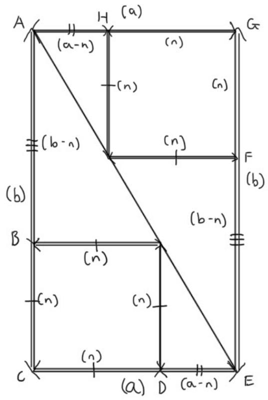
Ergo, using the general theorems for similar triangles, it can be understood that ΔBED and ΔECF are similar. This is due to the fact that $\angle BDE = \angle EFC$, $\angle BED = \angle ECF$, $\angle DBE = \angle FEC$, $FC \approx DE$ (which implies that $a - n \approx n$) and $EF \approx BD$ (which similarly suggests that $n \approx b - n$). After proving that the aforesaid triangles are indeed similar in nature, the following expressions (which are equal) can be formed: $\frac{n}{a-n}$ and $\frac{b-n}{n}$. Since these two expressions are equal, it would be suffice to say that the following formula is valid: $\frac{n}{a-n} = \frac{b-n}{n}$. Using the cross multiplication method, the equation can be simplified as follows: $n \cdot n = (a - n)(b - n)$, leading eventually to $n^2 = (a - n)(b - n)$.

Thus, expanding the brackets present on the right hand side of the formed equation leads to $n^2 = ab - an - bn + n \cdot n$, or $n^2 = ab - an - bn + n^2$, where 'n^2' is a common term on both sides of the formula. Cancelling this term, therefore would prove beneficial to find the value of n in algebraic terms. This endeavour leads to the given equation being expressed as $n^2 - n^2 = ab - an - bn$ or $0 = ab - an - bn$. This formula, afterwards, evolves into $an + bn = ab$ whereby 'n' is the highest common factor (also written as HCF) on the left hand side of the equation. Moreover, factorising 'n' out (on the left, that is) leaves us with $n(a + b) = ab$ so $\therefore n = \frac{ab}{a+b}$. This formula allows one to calculate the side length of the square, simply by dividing the sum of the length and base width of any right-angled triangle with the product (via multiplication) of the two preceding sides (in non-algebraic terms, that is). To test the formed equation (and ensure that its validity is not compromised), a duplet of values will be inputted into this algorithm. These values are '3' and '6' for 'a' and 'b' respectively. Such positive integer values in this formula leads to 'n' being visualised as $\frac{3 \cdot 6}{3+6}$ or $\frac{18}{9}$ which cancels to become $18 \div 9 = 2$. Henceforth, realising that $n < a$ and $n < b$ must be true such that $a - n > 0$, and that this condition is satisfied in this set of values (and others too, which will be shown later onwards), it was concluded that the formula must be true in nature. Whilst this approach was the first and foremost one formed by myself, there are two others that should most certainly be taken into account (which, at the end of the day, lead to the same formula).

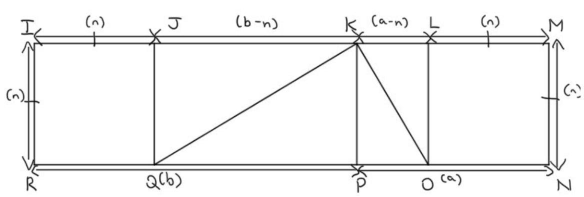


Using the labelled diagram above (applicable to the subsequent solution devised and explained), consider the following coordinates on a set of coordinate axes: $A = (0, 0)$ (which implies that this point is the origin), $B = (0, b)$, $C = (a, 0)$, $D = (0, x)$, $E = (x, x)$ and $F = (x, 0)$. The aforementioned mathematical scenario is whereby the distance between coordinates $(0, x)$ and (x, x) are numerically equal to the value of $'n'$. This has been presented as such to differentiate between the coordinates on the given plane and the lengths of the triangle formed. Accordingly, it can be deduced (using prior mathematical knowledge) that the equation of all lines which constitute the given shape are in the form of $y = mx + c$. This general system of equations applies in the case that the gradient (denoted as $'m'$ in the mentioned equation) is a constant coefficient of $'x'$ and the intercept on the y axis is singular in nature (denoted as c in the above-stated formula). $\therefore$, it shouldn't come as a surprise that in the straight line of BEC, the gradient (which is negative, principally because the slope is downwards) is $-\frac{b}{a}$ and the cut through point is b.

Using such constituents, the equation of line BEC would be $y =- \frac{b}{a}x + b$ (also written as $y = b - \frac{bx}{a}$). Simplifying the right hand side of the formula gives us $y = \frac{b}{1} - \frac{bx}{a} = \frac{a(b) - x(b)}{a} = \frac{b(a - x)}{a}$ or $y = \frac{ab - bx}{a}$ In the scenario where $y = x$ at $E = (x, x)$, the above equation can lead us to $x = \frac{ab - bx}{a}$, which in turn can be cross multiplied, leaving the formula of $ax = ab - bx$. $\therefore$ $ax + bx = ab$, where $'x'$ is the highest common factor, factorising to produce $x(a + b) = ab$. As mentioned previously, the distance $'x'$ = the value of $'n'$, implying that in reality, $n(a + b) = ab$. Thus, it would suffice to say that $n = \frac{ab}{a + b}$. To further evaluate that the formed equation is correct in nature, an entirely different duplet of values will be inputted. These said values are '4' and '12' for $'a'$ and $'b'$ respectively. Akin positive integer values in this formula leads to $'n'$ being pictured as $\frac{4 \cdot 12}{4 + 12}$ or $\frac{48}{16}$ which cancels to create $48 \div 16 = 3$. Henceforward, understanding that $n < a$ and $n < b$ must be true such that $a - n > 0$, whilst such statements are true in the given data set, it was evidently concluded that the formula is indeed correct. The succeeding methodology presented leads (once again) to the same equation, which suggests that, in theory, there are few reasons as to why the formula should be incorrect. Nonetheless, it is vital for myself to explain them such that students and teachers alike can openly critique them and find any potential flaws, which can be corrected in the future (regardless of when they arise).



Observing the labelled diagram seen above (relevant to the final and lattermost solution), consider the previously shown triangle, whereby it is used such that it is a constituent of the given rectangle. It would suffice to say that the combination of two such triangles to form such a polygon will not affect in any way possible the lengths and widths for the sides at hand. Henceforth, the following diagram can be drawn using the components of the polygon above:



Therefore, after carefully analysing both images above, a deducible (whilst being visually representable) fact would be that there are multiple common sides, and that the total area of both rectangles is equal in reality. Such an observation can be made from several facts (where sides noted are present in one of both diagrams drawn): $AH = DE = KL$ for the value of $(a - n)$ $HG = GF = BC = CD = JI = IR = LM = MN$ for the value of $'n'$ and $AG = CE = PN$ for the value of $'a'$.

Moreover, further facts to prove such a deduction (with reasoning which will be explained later onwards) include: $AB = FE = JK$ for the value of $(b - n)$ and $AC = GE = RP$. Accordingly, it shouldn't come as a surprise that $RP = IJ + JK$ (or, expressed in algebraic terms, $b = n + b - n$) and $PN = KL + LM$ (additionally expressed as $a = a - n + n$). This is the scenario whereby the value of $'n'$ cancels out on the right hand side of both equations. From this, another notable fact (that too, of significant importance) would be that since $RN = IM = n + b - n + a - n + n = a + b$, the area of $IMNR$ is $n(a + b)$. Likewise, a fact of similar nature would be that the area of $AGEC$ is $a * b = ab$. Thereby, it can be be concluded that where both such equations are equal (since the areas of both rectangles are equal as they have the same constituent polygons), $n(a + b) = ab$. $\therefore$, the final concluded formula as such is $n = \frac{ab}{a + b}$.

Conclusion

Following the representation of a trio of methodologies to solve the given problem, it can be ascertained that these systems of equations are valuable principally for their simplicity. Despite the relative complexity and detail present in the solution, the overall approach is fairly straightforward. In consequence, it is hoped by myself that such solutions do prove to be valuable to mathematicians, teachers and students alike.

Bibliography

➢ nrich.maths.org. The Square Under The Hypotenuse. [online] Available at: https://nrich.maths.org/11581