Irrational Roots

Quadratic polynomial with integer coefficients that has $x = 1 + \sqrt{2}$ as a root

If $x = 1 + \sqrt{2}$, substituting the numbers in the quadratic formula, assuming a = 1.

$\frac{2 \pm 2\sqrt{2}}{2} = 1 + \sqrt{(2)}$

$2\sqrt{2} = \sqrt{4 - 4c}$

$8 = 4 - 4c$

$-4c = 4$

$c = -1$

$a = 1, b = -2, c = -1$

$x^2 - 2x - 1$

Solutions of $x^2 + 2x - 1 = 0\ are 1 - \sqrt{2}\ and\ 1 + \sqrt{2}$.

Quadratic polynomial with integer coefficients that has $x = 1 + \sqrt{3}$ as a root

If $x = 1 + \sqrt{3}$, substituting the numbers within the quadratic formula, assuming a = 1

$\frac{2 \pm 2\sqrt{3}}{2} = 1 + \sqrt{3}$

$2\sqrt{3} = \sqrt{4 - 4c}$

$12 = 4 - 4c$

$-4c = 8$

$c = -2$

$a = 1, b = -2, \quad c = -2$

$x^2 - 2x - 2$

$x^2 + 2x - 2 = 0\ are 1 - \sqrt{3}\ and\ 1 + \sqrt{3}.$

Generalising $1 + \sqrt{n}$

We have found in both equations that a is always equal to 1, and that b is always equal to -2. The variable c seems to be $-n \quad + 1$,, as shown by $-2 + 1 = -1$

Therefore, the general formula is

$$x^2 - 2x - n + 1$$

Quadratic polynomial with integer coefficients that has $x = 2 + \sqrt{5}$ as a root (to help generalise for $m + \sqrt{n}$)

If $x = 2 + \sqrt{5}$, substituting the numbers within the quadratic formula, assuming a = 1

$\frac{4 \pm 2\sqrt{5}}{2} = 2 + \sqrt{5}$

$2\sqrt{5} = \sqrt{16 - 4c}$

$20 = 16 - 4c$

$4 = -4c$

$c = -1$

$a = 1, b = -4, c = -1$
$x^2 - 4x - 1$
$Solutions\ for\ x^2 - 4x - 1 = 0\ are\ 2 - \sqrt{5}, 2 + \sqrt{5}$

Generalising $m + \sqrt{n}$

We have found that a is always equal to 1, even when m is not equal to 1. Taking into account the fact that $1 + \sqrt{n}$ results in b = -2, and $2 + \sqrt{5}$ results in b = -4, $b = -2m$. To gain c, we simply take -n again, and add m^2 to it - for example, $-5 + 2^2 = -1$.

Therefore, the general formula is:

$$x^2 - 2mx - n + m^2$$

Testing with $6 + \sqrt{11}$

$$x^2 - 12x - 11 + 36$$
$$x^2 - 12x + 25$$

By completing the square:

$$(x - 6)^2 = x^2 - 12x + 36$$
$$(x - 6)^2 - 11 = 0$$
$$(x - 6)^2 = 11$$
$$x - 6 = \sqrt{11}$$
$$x = 6 \pm \sqrt{11}$$