

For some of these, I knew the answers straight away, but I wanted to prove it, so I went back to the rules of indices. Others were harder!

a) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

I know this works for all non negative integers: e.g. $a = 4, b = 9$

| LHS | RHS |
|----------------------------|---------------------|
| $\sqrt{4} \times \sqrt{9}$ | $\sqrt{4 \times 9}$ |
| $= 2 \times 3$ | $= \sqrt{36}$ |
| $= 6$ | $= 6$ |

$(a^m)^n = a^{mn}$ and $a^{(1/2)} = \sqrt{a}, \therefore \sqrt{a} = a^{(1/2)}, \sqrt{b} = b^{(1/2)}, \sqrt{ab} = (ab)^{(1/2)}$

Rephrase the expression using indices and square throughout

| LHS | RHS |
|--|---------------------------|
| $(a^{(1/2)} \times b^{(1/2)})^2$ | $((ab)^{(1/2)})^2$ |
| $= a^{(1/2) \times 2} \times b^{(1/2) \times 2}$ | $= (ab)^{(1/2) \times 2}$ |
| $= a^1 \times b^1$ | $= (ab)^1$ |
| $= (ab)^1$ | |

\therefore any non-negative value for a, b will work

b) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

This is the same as the first part, except that we are dividing instead of multiplying. Using the distributive law, calculate:

$$\frac{a \times a \times a \times a \times a \times a \times a \times \dots a}{b \times b \times b \times b \times b \times b \times b \times \dots b} \quad \begin{array}{l} a \text{ appears } m \text{ times in the numerator} \\ b \text{ appears } m \text{ times in the denominator} \end{array}$$

$$= \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \dots \left(\frac{a}{b}\right) \quad \left(\frac{a}{b}\right) \text{ appears } m \text{ times} \therefore \left(\frac{a}{b}\right)^m$$

E.g. $\frac{4^3}{2^3} = \left(\frac{4}{2}\right) \times \left(\frac{4}{2}\right) \times \left(\frac{4}{2}\right) = \left(\frac{4}{2}\right)^3$

Using this we can say: $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$ and we know $\sqrt{a} = a^{(1/2)}, \sqrt{b} = b^{(1/2)}$

$\therefore \frac{a^{(1/2)}}{b^{(1/2)}} = \left(\frac{a}{b}\right)^{(1/2)} \quad \therefore \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

This will work for all non negative values for a, but b cannot be 0, because division by 0 is undefined.
 \therefore b can only be a positive integer.

Proof: $\frac{100}{0} = x \quad \therefore 100 = 0x$ which is impossible: no multiple of 0 can equal 100 as $0x = 0$.

But: $\frac{0}{100} = x \quad \therefore 0 = 100x$ which is possible when $x = 0$.

c) This has three layers of surds, so I will come back to it later.

$$d) a \times \sqrt{b} = \sqrt{ab}$$

We are given:

From part (a) we know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, so substitute:
divide throughout by \sqrt{b}

| LHS | RHS |
|---------------------|----------------------------|
| $a \times \sqrt{b}$ | \sqrt{ab} |
| $a \times \sqrt{b}$ | $\sqrt{a} \times \sqrt{b}$ |
| a | \sqrt{a} |

There are only two values for a that work, 0 and 1, because any power where 1 is the base produces 1. Similarly any power where 0 is the base produces 0. The answer is $a = 0$, or $a = 1$, b can be any non negative value.

$$e) \frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}} = 1 \quad \therefore \text{in this question, } \sqrt{ab} = \sqrt{a} + \sqrt{b}$$

If $a = 0$ then $\frac{0}{\sqrt{b}} \neq 1$, similarly if $b = 0$ then $\frac{0}{\sqrt{a}} \neq 1$, because 0 divided by any number = 0

If $a = b = 0$, then $\frac{0}{0} \neq 1$ because division by 0 is undefined.

$$\therefore (a \neq 0), (b \neq 0), (a, b \neq 0)$$

If $a = b$, then:

$$\frac{\sqrt{a^2}}{2 \times \sqrt{a}} = 1$$

divide throughout by \sqrt{a} to get: $\frac{\sqrt{a}}{2} = 1 \quad \therefore \sqrt{a} = 2 \quad \therefore a = 4$

substitute $a = b = 4$ back into the equation to get:

$$\frac{\sqrt{4 \times 4}}{\sqrt{4} + \sqrt{4}} = 1$$

$\therefore a = 4, b = 4$ satisfies the criteria. This is unlikely to be the only solution, but it is likely to be the only solution where both a and b are integers.

To find any others I must come up with a rule. To do this I think I need to factorize, to find all solutions, but I spent ages trying to do so. Whenever I don't understand something, I wash some numbers through. I used square numbers to make it easier. I tried lots of examples, but these worked:

$$\frac{3^2}{2^2}, \frac{4^2}{3^2}, \frac{5^2}{4^2}, \text{ etc ...}$$

| These give: | Numerator: | Denominator |
|-------------|---|--|
| | $\frac{\sqrt{3^2}}{\sqrt{2^2}} \times \frac{\sqrt{3^2}}{1} = \frac{3}{2} \times \frac{3}{1} = \frac{9}{2}$ | $\frac{\sqrt{3^2}}{\sqrt{2^2}} + \frac{\sqrt{3^2}}{1} = \frac{3}{2} + \frac{6}{2} = \frac{9}{2}$ |
| | $\frac{\sqrt{4^2}}{\sqrt{3^2}} \times \frac{\sqrt{4^2}}{1} = \frac{4}{3} \times \frac{4}{1} = \frac{16}{3}$ | $\frac{\sqrt{4^2}}{\sqrt{3^2}} + \frac{\sqrt{4^2}}{1} = \frac{4}{3} + \frac{12}{3} = \frac{16}{3}$ |
| | $\frac{\sqrt{5^2}}{\sqrt{4^2}} \times \frac{\sqrt{5^2}}{1} = \frac{5}{4} \times \frac{5}{1} = \frac{25}{4}$ | $\frac{\sqrt{5^2}}{\sqrt{4^2}} + \frac{\sqrt{5^2}}{1} = \frac{5}{4} + \frac{20}{4} = \frac{25}{4}$ |

This series works, and I predict that it will continue to do so.

I can write this algebraically as:

$$\frac{\left(\frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} \times \sqrt{n^2}\right)}{\left(\frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} + \sqrt{n^2}\right)}$$

To prove that this works, simplify:

Numerator:

$$\begin{aligned} & \frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} \times \sqrt{n^2} \\ &= \frac{n}{(n-1)} \times \frac{n}{1} \\ &= \frac{n^2}{(n-1)} \end{aligned}$$

Denominator:

$$\begin{aligned} & \frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} + \sqrt{n^2} \\ &= \frac{n}{(n-1)} + \frac{n}{1} \\ &= \frac{n}{(n-1)} + \frac{n^2 - n}{(n-1)} \\ &= \frac{n^2}{(n-1)} \end{aligned}$$

Numerator = denominator \therefore quotient must always be 1 \therefore there are infinite non-negative values for a, b when:

$a = \frac{\sqrt{n^2}}{\sqrt{(n-1)^2}}$, $b = \sqrt{n^2}$ or vice versa, where n is a positive integer greater than 1.

f) $\sqrt{a} - \sqrt{b} = \sqrt{a-b}$

| | LHS | RHS |
|-------------------------------|--------------------------------|----------------------|
| We are given: | $\sqrt{a} - \sqrt{b}$ | $\sqrt{a-b}$ |
| Start by squaring both sides: | $a - (2 \times \sqrt{ab}) + b$ | $a - b$ |
| Subtract a from both sides: | $b - (2 \times \sqrt{ab})$ | $-b$ |
| Rearrange: | $2b$ | $2 \times \sqrt{ab}$ |
| Simplify: | b | \sqrt{ab} |
| Square both sides to get: | b^2 | ab |
| Divide both sides by b : | b | a |

For this equation to be true $a = b$

Check:

Going back to the original equation, if $a = b$, then it will work BUT, there is another possibility also.

As any number minus zero remains unchanged a can be any non negative value when $b = 0$. However, when $a = 0$, the result contains the square root of a negative number, which is not allowed (we are looking for non negative values, not imaginary numbers).

g) has several layers, so I'll leave it until later.

$$h) \frac{\sqrt{a+b}}{\sqrt{c+d}} = (\sqrt{a} + b)(\sqrt{c} - d)$$

| | LHS | RHS |
|---------------|-------------------------------------|--------------------------------|
| We are given: | $\frac{\sqrt{a} + b}{\sqrt{c} + d}$ | $(\sqrt{a} + b)(\sqrt{c} - d)$ |
| Rearrange: | $\frac{\sqrt{a} + b}{\sqrt{a} + b}$ | $(\sqrt{c} + d)(\sqrt{c} - d)$ |
| Simplify: | 1 | $c - d^2$ |
| Rearrange: | $d^2 + 1$ | c |

In the original statement $a = b = 0$ is permitted as both sides of the equation would be 0, but $c = d = 0$ is not permitted, as division by 0 is undefined.

\therefore the statement is true when either $c = d^2 + 1$ or $a = b = 0$, but $c = d \neq 0$

$$i) \sqrt{5 + (2 \times \sqrt{6})} = \sqrt{a} + \sqrt{b}$$

| | LHS | RHS |
|-------------------|----------------------------------|----------------------------------|
| We are given: | $\sqrt{5 + (2 \times \sqrt{6})}$ | $\sqrt{a} + \sqrt{b}$ |
| Square each side: | $5 + (2 \times \sqrt{6})$ | $a + (2 \times \sqrt{ab}) + b$ |
| Rearrange: | $5 + (2 \times \sqrt{6})$ | $(a + b) + (2 \times \sqrt{ab})$ |

It's logical to suppose that $(a + b) = 5$, and $ab = 6$ Factors of 6 are (1x6) (2x3) Numbers which sum to 5 are (0+5) (1+4) (2+3) 2,3 are the only numbers that appear in both sets, $\therefore \{a, b\} = \{2, 3\}$ in any order

$$g) \sqrt{a} + \sqrt{b} = \sqrt{a + b + \sqrt{4ab}}$$

| | LHS | RHS |
|---------------------------------|--------------------------------|-----------------------------|
| We are given: | $\sqrt{a} + \sqrt{b}$ | $\sqrt{a + b + \sqrt{4ab}}$ |
| Square each side: | $a + (2 \times \sqrt{ab}) + b$ | $a + b + \sqrt{4ab}$ |
| Subtract $a + b$ from each side | $2 \times \sqrt{ab}$ | $\sqrt{4ab}$ |
| Simplify | $2 \times \sqrt{ab}$ | $2 \times \sqrt{ab}$ |

$\therefore a, b$ can be any non-negative value for the equation to work!

$$c) \sqrt{23 - (6 \times \sqrt{6 - (4 \times \sqrt{2})})} = \sqrt{a} + \sqrt{b}$$

This was quite difficult and I had to walk away for a while.

The numbers on the LHS simplify to $3 + \sqrt{2} \therefore \sqrt{a} + \sqrt{b} = \sqrt{9} + \sqrt{2}$, or vice versa

But this isn't a solution, so I worked backwards to try and unravel the algebra and prove the result:

LHS

RHS

We are given:

$$\sqrt{23 - \left(6 \times \sqrt{6 - (4 \times \sqrt{2})}\right)}$$

$$\sqrt{a} + \sqrt{b}$$

Square both sides:

$$23 - \left(6 \times \sqrt{6 - (4 \times \sqrt{2})}\right)$$

$$(\sqrt{a} + \sqrt{b})^2$$

Rearrange:

$$23 - (\sqrt{a} + \sqrt{b})^2$$

$$\left(6 \times \sqrt{6 - (4 \times \sqrt{2})}\right)$$

Divide throughout by 6:

$$\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}$$

$$\sqrt{6 - (4 \times \sqrt{2})}$$

Square both sides

$$\left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2$$

$$6 - (4 \times \sqrt{2})$$

Rearrange:

$$4 \times \sqrt{2}$$

$$6 - \left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2$$

Divide throughout by 4:

$$\sqrt{2}$$

$$\frac{6 - \left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2}{4}$$

Add $\sqrt{9}$ to both sides:

$$\sqrt{9} + \sqrt{2}$$

$$\sqrt{9} + \left(\frac{6 - \left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2}{4}\right)$$

Now insert $a = 9, b = 2$
into RHS and simplify:

$$= 3 + \sqrt{2}$$

$$3 + \left(\frac{6 - \left(\frac{23 - (\sqrt{9} + \sqrt{2})^2}{6}\right)^2}{4}\right)$$

NOTE:

$$(\sqrt{9} + \sqrt{2})^2 = 11 + (6 \times \sqrt{2})$$

$$= 3 + \left(\frac{6 - (2 + \sqrt{2})^2}{4}\right)$$

$$(2 + \sqrt{2})^2 = 6 + (4 \times \sqrt{2})$$

$$= 3 + \left(\frac{4 \times \sqrt{2}}{4}\right)$$

$$= 3 + \sqrt{2}$$

Phew! This last question was really hard!

Also, as addition is commutative, $b = 9, a = 2$ is also possible. Brilliant question!