A cubic polynomial can be expressed in the form:

$$Ax^3 + Bx^2 + Cx + D$$

(Where A, B, C and D are real numbers. Note: I have used capital letters here in order to avoid confusion with the next part of my solution).

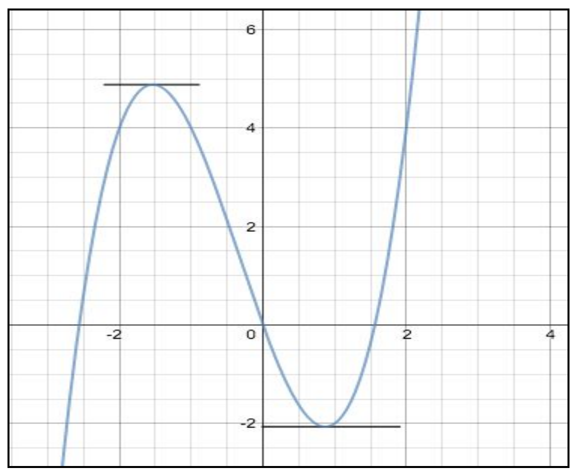
The gradient function $(dy\,/\,dx)$ of the generalised cubic expression shown above is as follows:

$$3Ax^2 + 2Bx + C$$

This is a quadratic equation which tells us the gradient of the cubic at a given x coordinate when we plug said x-coordinate into the quadratic equation (the result is the gradient of the cubic).

A stationary point is defined as a point on the curve where the gradient is 0, or where the gradient function of the curve is equal to 0.

This is what gives a cubic equation 'its curves' since when the gradient is 0, the part of the curve prior to the stationary point could be decreasing (i.e. have a negative gradient), and as we get closer to the stationary point, the gradient converges to zero before becoming positive once we go past the stationary point (and vice versa).



However, the stationary point could be a point of inflection; i.e. the gradients before and after the stationary point are both positive or they are both negative (therefore the cubic will always be increasing or decreasing and hence it is not curvy, assuming 'curvy' means having stationary points that behave in the manner described in the previous paragraph). The diagram on the right shows a 'curvy' cubic with tangents to the stationary points as the gradient goes from -ve to 0 to +ve and vice versa.

Since the gradient function of a cubic is a quadratic, there are a maximum of two stationary points and a minimum of zero (since $ax^2 + bx + c = 0$ has a maximum of two real solutions and a minimum of zero real solutions).

For a cubic to be 'curvy', the gradient function (i.e the quadratic) must cross the x-axis (as this is where the gradient changes from +ve to -ve and vice versa); therefore the quadratic must have two real solutions when set to equal zero (since having none will mean the quadratic stays +ve or -ve and having one will mean the quadratic touches the

x-axis, but does not change sign, i.e. a repeated root) meaning there are two stationary points.

Therefore, the conditions for a curvy cubic are as follows:

A general cubic equation's gradient function must have two real solutions when set to equal zero.

$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, b^2-4ac **must be +ve for the quadratic to yield two different real solutions (if it were -ve, then x would equal two complex numbers as you cannot square root a negative number to get a real solution, therefore the quadratic won't intercept the x-axis as the standard Cartesian plane deals with real numbers only. If** b^2-4ac **was equal to zero, x would have only one solution, namely** $-b/2a$ **as this is what one gets when** $b^2-4ac=0$ **is substituted into the quadratic formula).**

b^2 corresponds to '$4B^2$' in the general cubic gradient function $3Ax^2+2Bx+C$ **and -4ac corresponds to -12AC.**

Therefore, for a cubic in the form Ax^3+Bx^2+Cx+D **to be 'curvy:'**

$$4B^2-12AC>0$$

The converse of this is an 'uncurvy graph' (like the one shown in the question) where the gradient is never positive or negative (i.e. the graph is always decreasing or increasing respectively).

Therefore the gradient function must never cross the x-axis (therefore the cubic will not have any stationary points) or it must touch it, but not change sign (i.e. a repeated root), which corresponds to the cubic having one stationary point, but the gradient stays either positive or negative before and after the stationary point (this is called a point of inflection).

Therefore, for a general quadratic. $b^2-4ac\leq 0$

As before, b^2 corresponds to '$4B^2$' in the general cubic gradient function $3Ax^2+2Bx+C$ **and -4ac corresponds to -12AC.**

Therefore, to generate an ‘uncurvy’ cubic (like the one shown in the question) where the gradient stays +ve or -ve and doesn’t change between the two:

$$4B^2 - 12AC \leq 0$$

When the cubic is expressed in the form: $Ax^3 + Bx^2 + Cx + D$