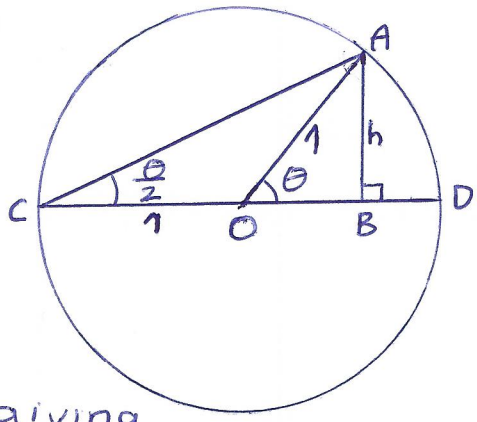
Question:

To prove $\cos^2\frac{\theta}{2} \equiv \frac{1}{2}(1+\cos\theta)$



We can draw a circle with radius 1 and an angle of θ as shown.
$\angle AOB = \theta$ and $\angle ACB = \frac{\theta}{2}$

The line AB is perpendicular to CD giving us two right angled triangles, ΔABC and ΔAOB
Let $AB = h$

In ΔAOB: $OB = \sqrt{1-h^2}$ (by using Pythagoras Theorem)

$$\cos\theta = \frac{OB}{AO} = \frac{\sqrt{1-h^2}}{1} = \sqrt{1-h^2}$$
$$\Rightarrow \cos^2\theta = 1-h^2$$
$$\Rightarrow h^2 = 1-\cos^2\theta$$

In ΔABC: $BC = BO + OC = 1 + \sqrt{1+h^2}$

$$BC^2 = (1+\sqrt{1+h^2})^2$$
$$AB^2 = h^2$$
$$\Rightarrow AC^2 = (1+\sqrt{1+h^2})^2 + h^2$$
$$\cos^2\frac{\theta}{2} = \frac{BC^2}{AC^2} = \frac{(1+\sqrt{1+h^2})^2}{(1+\sqrt{1+h^2})^2+h^2}$$

Since we know how $\cos\theta = \sqrt{1-h^2}$ and $h^2 = 1-\cos^2\theta$, we can get:

$$\cos^2\frac{\theta}{2} = \frac{(1+\cos\theta)^2}{(1+\cos\theta)^2+1-\cos^2\theta}$$
$$= \frac{(1+\cos\theta)^2}{1+\cos^2\theta+2\cos\theta+1-\cos^2\theta}$$
$$= \frac{(1+\cos\theta)^2}{2+2\cos\theta}$$
$$= \frac{1+\cos\theta}{2}$$

So: $\cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$

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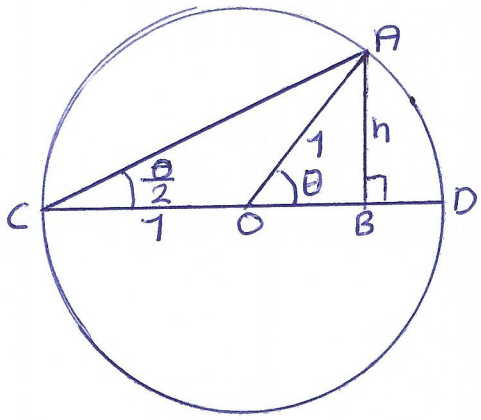
Question:

To prove $\sin^2\frac{\theta}{2} \equiv \frac{1}{2}(1-\cos\theta)$



We can use the same circle to prove the Sine Half Angle Formula.

In ΔAOB: $\sin\theta = \frac{AB}{AO} = \frac{h}{1} = h$

In ΔABC: $\sin\frac{\theta}{2} = \frac{h}{\sqrt{(1+\sqrt{1-h^2})^2+h^2}}$

$$\sin^2\frac{\theta}{2} = \frac{h^2}{(1+\sqrt{1-h^2})^2+h^2}$$

Since $\cos\theta = \sqrt{1-h^2}$ and $h^2 = 1-\cos^2\theta$, we can get:

$$\sin^2\frac{\theta}{2} = \frac{1-\cos^2\theta}{(1+\cos\theta)^2+1-\cos^2\theta}$$

$$= \frac{1-\cos^2\theta}{2+2\cos\theta}$$

$$= \frac{(1+\cos\theta)(1-\cos\theta)}{2(1+\cos\theta)}$$

$$= \frac{1-\cos\theta}{2}$$

So: $\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$

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