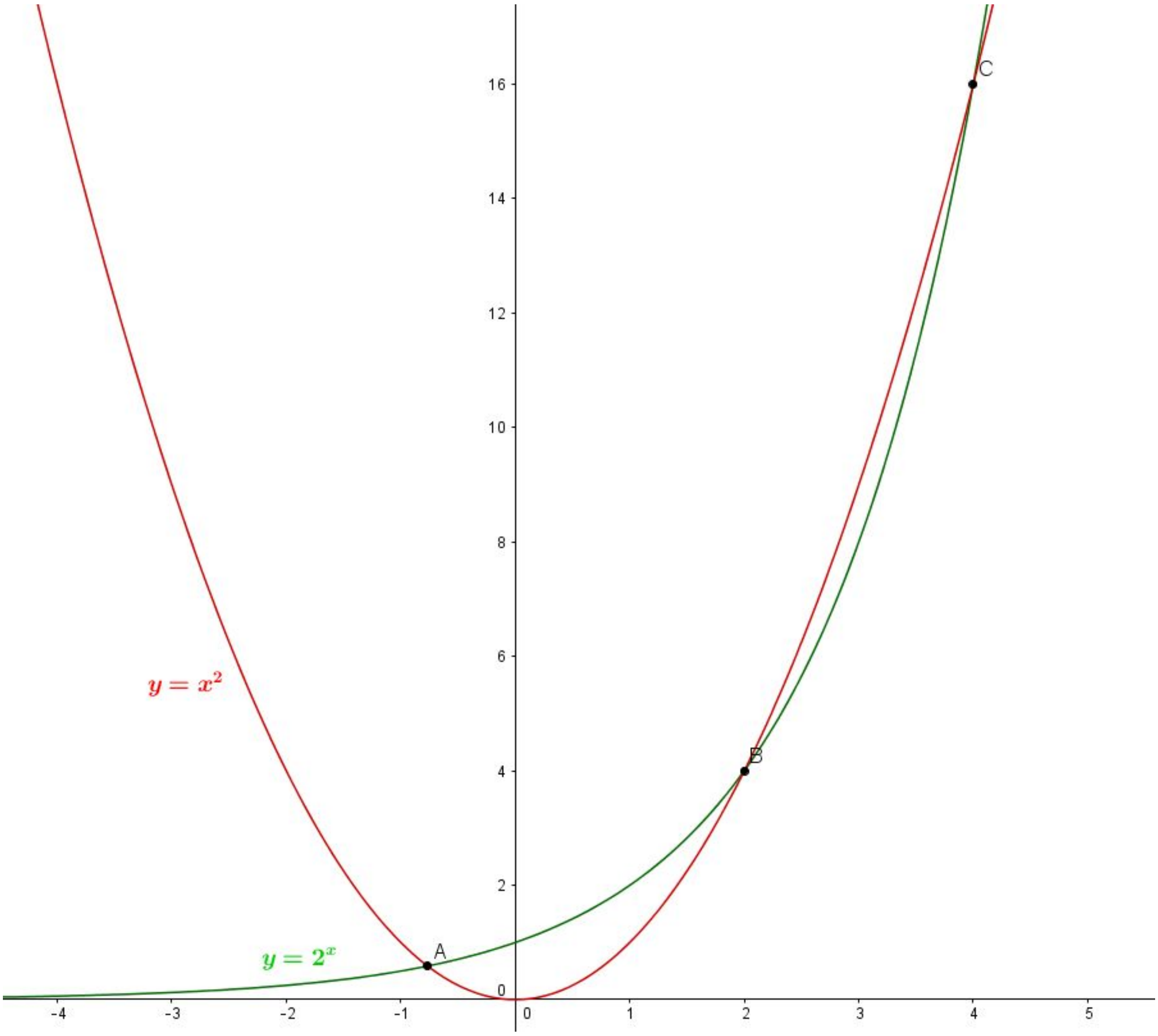
Original question:

Which is bigger:

- 2^x or x^2?
- a^x or x^a?

(http://nrich.maths.org/12253)

If we draw a graph of the two functions:



x < A	A < x < B	B < x < C	C < x
x^2 is bigger	2^x is bigger	x^2 is bigger	2^x is bigger

To be able to state when 2^x is bigger or when x^2 is bigger, we need to know the coordinates of the points of intersection. So when is $2^x = x^2$? My first thought was to use logs:

$$2^x = x^2$$
$$log(2^x) = log(x^2)$$
$$xlog(2) = 2log(x)$$
$$\frac{x}{log(x)} = \frac{2}{log(2)}$$

But I was left with x and log(x) stuck to together with no way to separate them. Furthermore, I realised that it would only give me positive solutions, as log(x) only exists when x>0. This would only give me two out of the three intersections.

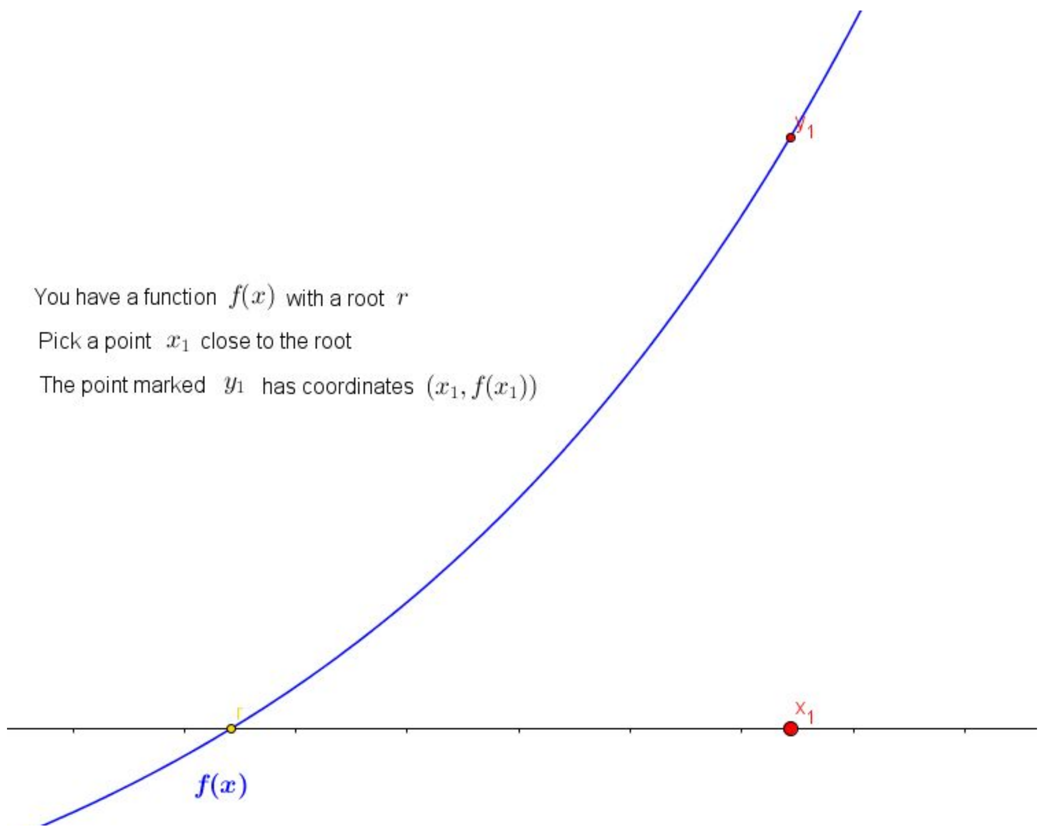
To find the solution, I could use a graphing program to find the points of intersection but if I wanted to generalise it to a^x and x^a it would be impractical. I decided to look for other methods online to find the points of intersection.

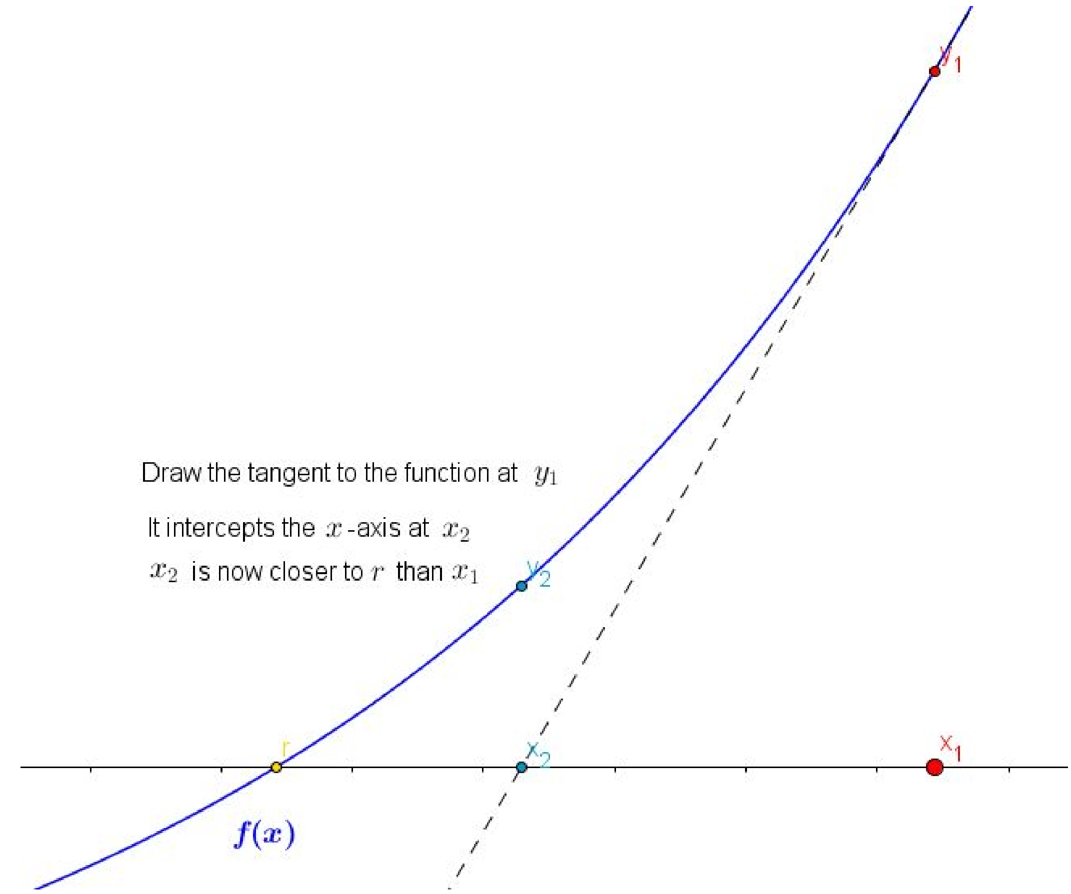
<u>The Newton-Raphson Method</u>

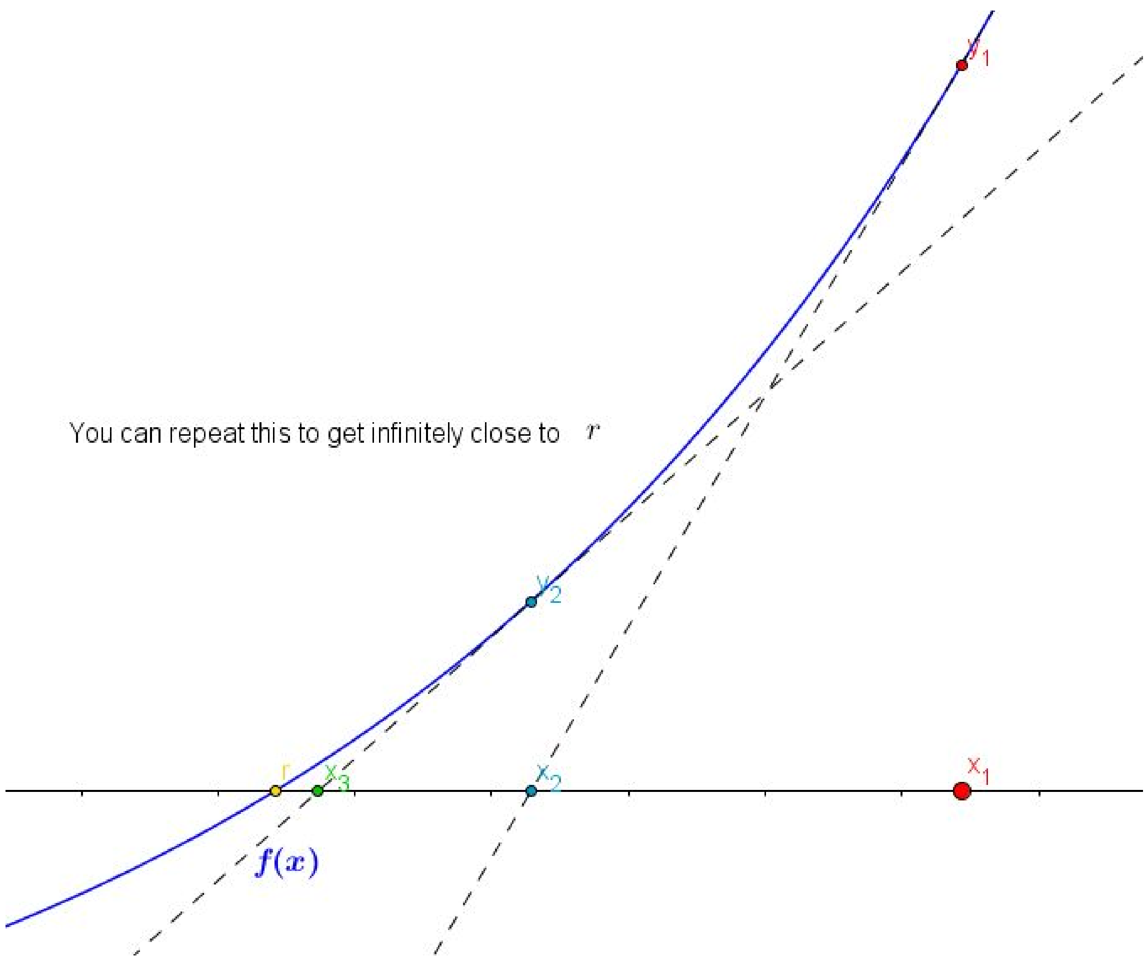
The Newton-Raphson Method is a way of finding the roots of functions by constantly repeating a formula (iteration).

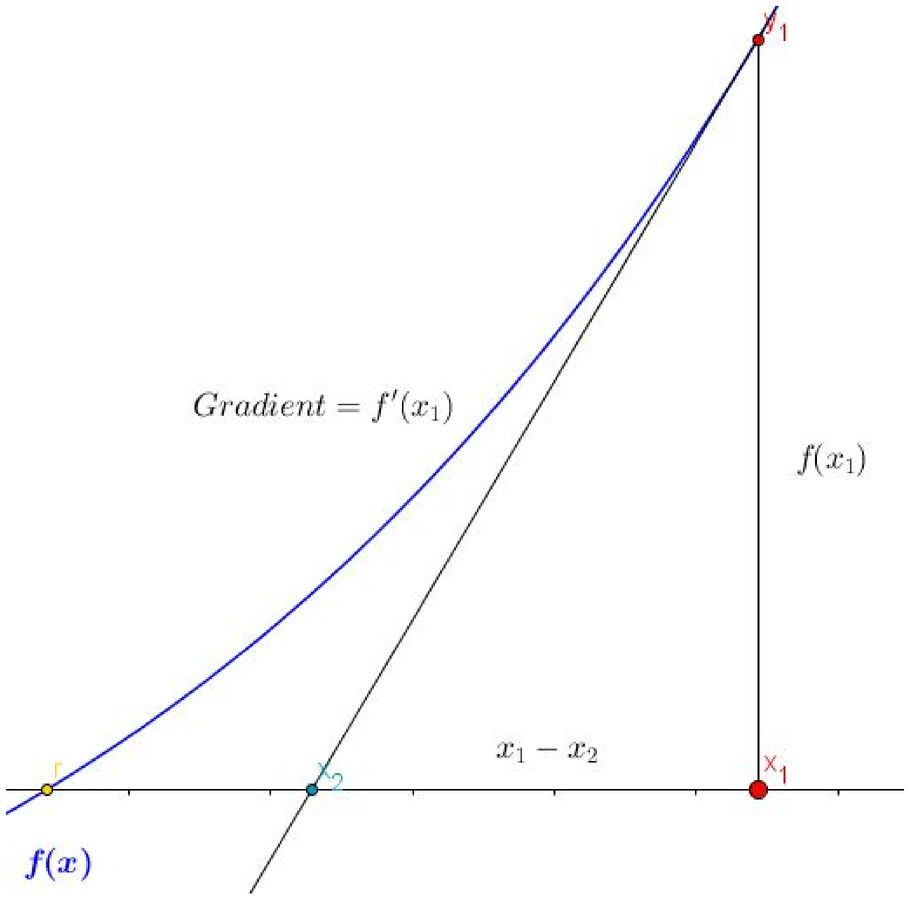
The formula $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ is:

How it works:




Draw the tangent to the function at y_1
It intercepts the x-axis at x_2
x_2 is now closer to r than x_1
y_1
y_2
r
x_2
x_1
$f(x)$


You can repeat this to get infinitely close to r
y_1
y_2
r
x_3
x_2
x_1
$f(x)$



$$f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

The gradient of the line at y_1 is f ' (x_1) by definition.
And then using

$$m = \frac{rise}{run}$$

$$x_1 - x_2 = \frac{f(x_1)}{f'(x_1)}$$

Rearranging…

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Gives us the formula!

So you can find the root of a function by iterating the formula. Let's try it for $f(x) = 2^x - x^2$

The derivative of that function is: $f'(x) = 2^x ln(2) - 2x$

The fraction would be: $\frac{2^x - x^2}{2^x ln(2) - 2x}$

Since we want to find the value of the intersection with negative x coordinate, let's pick x_1= -1
If we put this into our iterative formula:

$$x_2 = -1 - \frac{f(-1)}{f'(-1)}$$

$$x_2 = -0.7869233$$
$$x_3 = ...$$
$$x_4 = ...$$
$$x_5 = -0.76666469$$

This is very close to one of the roots $f(x) = 2^x - x^2$ of

If we choose different starting values for x_1 , you get different roots:
x_1= 1 then the root = 2
x_1 =5 then the root = 4

If the roots of the equation are -0.76666, 2 and 4 then:

$$2^2 - 2^2 = 0 \Rightarrow 2^2 = 2^2$$
$$2^4 - 4^2 = 0 \Rightarrow 2^4 = 4^2$$
$$2^{-0.76666} - (-0.76666)^2 = 0 \Rightarrow 2^{-0.76666} = (-0.76666)^2$$

So 2^x and x^2 intersect at three points. Those points are:

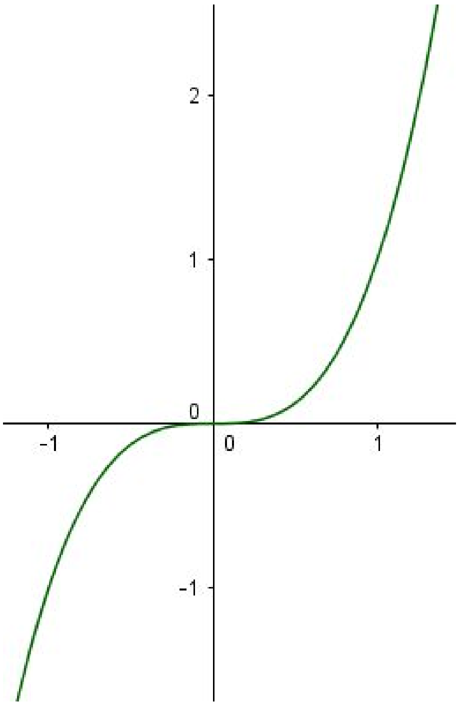
$(2 , 2^2)$ = **(2 , 4)**
$(4 , 2^4)$ = **(4 , 16)**
$(-0.76666 , 2^{-0.76666})$ = **(-0.76666, 0.58776)**

x < -0.766666	-0.76666 < x < 2	2 < x < 4	4 < x
x^2 is bigger	2^x is bigger	x^2 is bigger	2^x is bigger

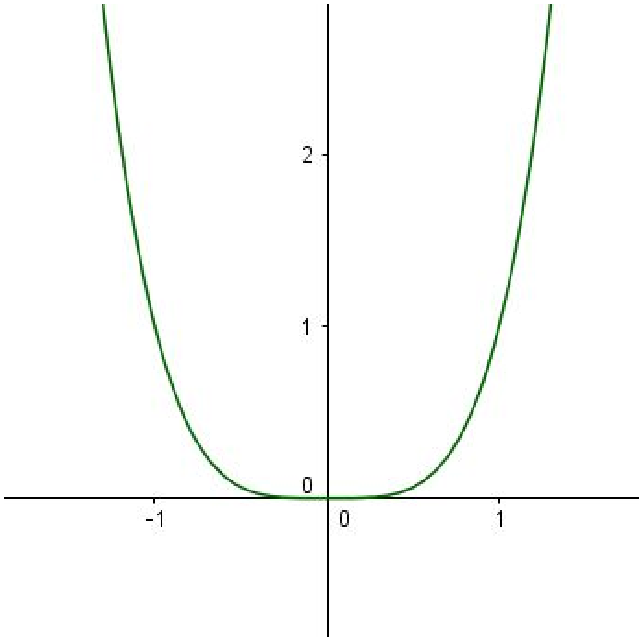
Generalising:

There a few problems with generalising this to a^x and x^a. Firstly, the Newton-Raphson Method converges slowly so it would take tens, maybe even hundreds, of iterations to get to a good level of accuracy. Plus if the initial guess, x_1, is very far away from the root it will not converge or it will take an impractical number of iterations to get the root.

Thirdly, not all graphs in the form x^a look the same:



Graph of x^3



Graph of x^4

Odd values of a give graphs that exist in the I and III quadrants. This means that it will not cross 3^x in the II quadrant. Even values of a give graphs that exist in the I and II quadrants, so they will intersect in the II quadrant.

However, we can try to generalise it for $f(x) = a^x - x^a$

Its derivative will be $f'(x) = a^x ln(a) - ax^{a-1}$

So if we input everything into the formula:

$$x_{n+1} = x_n - \frac{a^{(x_n)} - (x_n)^a}{a^{(x_n)} ln(a) - a(x_n)^{a-1}}$$

So what starting value do we pick for x_1?

EVEN VALUES OF a

Think how the graph of x^{200} would look like. The moment the x value passes 1, the graph shoots up into the sky. The same thing happens when the x value passes -1.

Now imagine what the graph of 200^x would look like. When x=0, y=1. Then it slowly but steadily grows until it reaches 1. At this point, y=200.

Using a calculator we can see that $200^{1.03} = 234.45…$ and that $1.03^{200} = 369.35…$
These numbers are very close together! So they probably intersect close to 1.03

Earlier we found out that 2^x and x^2 intersect at (2,4). And it seems that 200^x and x^{200} intersect somewhere very close to 1.03. From this we can deduce that when a is very large, the point of intersection will be at x=1. So for very large values of a (we're talking about 200), we can use $x_1 = 1.1$ as our starting value! (NOTE: the value is 1.1 because if the starting value we use is less than the root, it will converge to another root that is less than that, therefore, we use 1.1 because it is larger than the root for very large values of a)

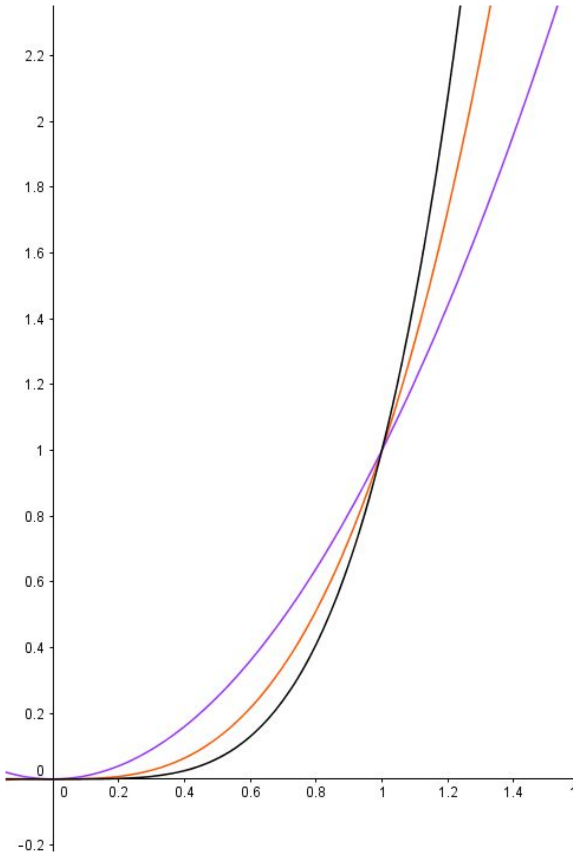
The second point of intersection in the I quadrant is very simple to find out:
When is $x^a = a^x$? One of the points of intersection will be when x = a

The third point is a little harder to get. If I repeat the method I used to get the roots of $2^x - x^2 = 0$ with a=10, I eventually got that the third root was at x=-0.83… The third root of $2^x - x^2 = 0$ was at x=-0.76666… So it seems that as a increases the point of intersection will get closer to x = -1

So for even values of a, the values of x_1 that we should use are $x_1 = 1.1$ and $x_1 = -0.9$ (for the same reason as before). The last root we know exactly, as it is at x = a.

ODD VALUES OF a

These graphs will only have two intersections, one if them at x = a.



These are the graphs of $y=x^2$ (purple), $y=x^3$ (orange) and $y=x^4$ (dark blue)

As you can see the graph of x^3 look a lot like x^2. Therefore we can assume they will act similarly.

So the value of x_1 that we should try is $x_1 = 1.1$, like with even values of a.

It won't intersect in the II quadrant.

Summarising

To approximate the solutions of the $f(x) = a^x - x^a$ function

We can use this iterative formula $x_{n+1} = x_n - \frac{a^{(x_n)} - (x_n)^a}{a^{(x_n)} ln(a) - a(x_n)^{a-1}}$

The starting values of x_1 we should use are x_1= 1.1 and x_1=-0.9 (the last one only for even values of a). The last point of intersection if when x = a.

We can use this to find out when a^x and x^a intersect so we can define when a^x is bigger or x^a is bigger. We could draw a table similar to the one we used for 2^x and x^2.

Example

Let's say you want to find when $300^x > x^{300}$. In this case, a = 300 The iterative equation is:

The value of x_1 you should try is $x_1 = 1.1$
$x_1 = 1.1$
$x_2 = 1.096333333$
$x_3 = 1.092678889$
$x_4 = 1.089036626$
$x_5 = 1.085406504$
$x_6 = 1.081788482$
$x_7 = 1.078182521$
$x_8 = 1.074588579$
… (as I said, slow to converge)
$x_{20} = 1.03243568$
$x_{21} = 1.029078738$
$x_{22} =1.025868197$
$x_{23} = 1.022998853$
…
$x_{27} = 1.019573863$
$x_{28} = 1.019573863$

We're done!

And now for x_1=-0.9
(I won't write everything down)

x_1 =
x_{29}= -0.9838222201
x_{30} = -0.9821566754
x_{31} = -0.9815703692
x_{32} = -0.9815124458
x_{33} = -0.9815119384
x_{34} = -0.9815119383
x_{35} = -0.9815119383

We're done!

Last root, x = 300

x < -0.9815	-0.9815 < x < 1.0196	1.0196 < x < 300	300 < x
x^{300} is bigger	300^x is bigger	x^{300} is bigger	300^x is bigger

This is hard to check, even with a graphing program, as 300^{300} is about 750 digits long!