Digital Equation

Part 1

i) $k,\ 0 \le k \le 9$

$10k - k^2$

$= -((k-5)^2 - 25)$

$= -(k-5)^2 + 25 \le 25$

$-k^2 + 10k - 25 + 25 \le 25$

$-k^2 + 10k \le 25$

$k = 0$

so $0 \le -k^2 + 10k \le 25$

ii) $k^3 - k$

$= k(k^2 - 1) \Rightarrow$ DOTS

$k(k-1)(k+1)$

$k-1, k, k+1$ are consecutive integers

It follows that one must be a multiple of 3.

Since 3 will be a factor of $k^3 - k$, this will be divisible by 3 for all real k.

Part 2

$N = 100a + 10b + c$

$S = a + b^2 + c^3$

$N = S$

$100a + 10b + c = a + b^2 + c^3$

$99a = (b^2 - 10b) + (c^3 - c)$

$99a - (b^2 - 10b) = c^3 - c$

Since $\begin{cases} 99a \text{ is a multiple of } 3 \\ c^3 - c \text{ is a multiple of } 3 \text{ (part 1)} \end{cases}$

$\therefore -b^2 + 10b$ must be a multiple of 3.

let $z = -b^2 + 10b$, $0 \leq b \leq 9$

we know from part 1 that the max of this is 25 and min is 0.

$b = 0$, $z = 0$
$b = 1$, $z = 9$ $\Rightarrow \times 3$
$b = 2$, $z = 16$
$b = 3$, $z = 21$ $\Rightarrow \times 3$
$b = 4$, $z = 24$ $\Rightarrow \times 3$
$b = 5$, $z = 25$
$b = 6$, $z = 24$ $\Rightarrow \times 3$
$b = 7$, $z = 21$ $\Rightarrow \times 3$
$b = 8$, $z = 16$
$b = 9$, $z = 9$ $\Rightarrow \times 3$

possible values of $-b^2 + 10b$ can only be $\{9, 21 \text{ or } 24\}$.

$c^3 - c$, for real c

$c = 5$, $= 120$
$c = 6$, $= 210$
$c = 7$, $= 336$
$c = 8$, $= 504$
$c = 9$, $= 720$.

possible value of $c^3 - c$ can only be $\{120, 210, 504, 720\}$

$99a$, for real a;

$= \{99, 198, 297, 396, 495, 594, 693, 792, 891, 990\}$.

↳ These are the possible values of $99a$

$99a = (c^3 - c) - (-b^2 + 10)$

Since we have all possible values of $c^3 - c$ and $-b^2 + 10b$, if they subtract to a multiple of 99, we have a solution for a, b, c respectively.

$120 - 9 = 111$, $\boxed{120 - 21 = 99}$, $120 - 24 = 96$

$210 - 9 = 201$, $210 - 21 = 189$, $210 - 24 = 186$

$336 - 9 = 327$, $336 - 21 = 315$, $336 - 24 = 312$

$\boxed{504 - 9 = 495}$, $504 - 21 = 483$, $504 - 24 = 480$

$720 - 9 = 711$, $720 - 21 = 699$, $720 - 24 = 696$

CASE 1:

$120 - 21 = 99$

$c^3 - c = 120$, $c = 6$

$-b^2 + 10b = 21$

$b^2 - 10b + 21 = 0$

$(b-7)(b-3)$

$b = 7$ $b = 3$

$99 = 99a$

$a = 1$

so $N = 135$ and 175

CASE 2:

$$504-9=495$$

$$c^3-c=504, c=8$$

$$-b^2+10b=9$$

$$b^2-10b+9=0$$

$$(b-9)(b-1)=0$$

$$b=9 \quad b=1$$

$$495=99a$$

$$a=5$$

$$N=518 \text{ and } 598$$

We can confirm there are no other solutions since the highest value of the LHS in $720-9$ $=711$. This is when $c=9$ and $b=1$ or 9.

Therefore the only solutions occurs when

$$N=135, 175, 518 \text{ and } 598$$