

Clock Arithmetic

Mahdi Raza Khunt
Mahatma Gandhi International School

Task 1

- What time is 5 hours after 10 o'clock?

$$10 + 5 = 15 = 3 + 12 \equiv 3 \pmod{12} \Rightarrow 3 \text{ o'clock}$$

- What time is 9 hours after 4 o'clock?

$$9 + 4 = 13 = 1 + 12 \equiv 1 \pmod{12} \Rightarrow 1 \text{ o'clock}$$

- What time is 13 hours after 6 o'clock?

$$13 + 6 = 19 = 7 + 12 \equiv 7 \pmod{12} \Rightarrow 7 \text{ o'clock}$$

- What time is 20 hours after 5 o'clock?

$$20 + 5 = 25 = 1 + 2 \times 12 \equiv 1 \pmod{12} \Rightarrow 1 \text{ o'clock}$$

- What time is 36 hours after 2 o'clock?

$$36 + 2 = 2 + 3 \times 12 \equiv 2 \pmod{12} \Rightarrow 2 \text{ o'clock}$$

Task 2

$$2 + 3 = 5 = 5 + 0 \times 7 \equiv 5 \pmod{7}$$

$$5 + 2 = 7 = 0 + 1 \times 7 \equiv 0 \pmod{7}$$

$$6 + 6 = 12 = 5 + 1 \times 7 \equiv 5 \pmod{7}$$

$$4 \times 2 = 8 = 1 + 1 \times 7 \equiv 1 \pmod{7}$$

$$3 \times 3 = 9 = 2 + 1 \times 7 \equiv 2 \pmod{7}$$

Task 3

$$2 - 5 = -3 = 4 - 1 \times 7 \equiv 4 \pmod{7}$$

$$1 - 2 = -1 = 6 - 1 \times 7 \equiv 6 \pmod{7}$$

Task 4

- 1 anticlockwise = + 6 clockwise
- 3 anticlockwise = + 4 clockwise
- 5 anticlockwise = + 2 clockwise
- 6 anticlockwise = + 1 clockwise

Task 5

$$20 = 6 + 2 \times 7 \equiv 6 \pmod{7}$$

$$32 = 4 + 4 \times 7 \equiv 4 \pmod{7}$$

$$50 = 1 + 7 \times 7 \equiv 1 \pmod{7}$$

$$91 = 0 + 13 \times 7 \equiv 13 \pmod{7}$$

Numbers of the form $x \equiv 3 \pmod{7}$ can be generated as such:

$$3 + 0 \times 7$$

$$3 + 1 \times 7$$

$$3 + 2 \times 7$$

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These form an Arithmetic progression with common difference as 7 and starting term as 3.

Thus, the general term is $3 + (n - 1)7 = 7n - 4$

Similarly, for $x \equiv a \pmod{7}$, the general term is $a + (n - 1)7 = 7n + (a - 7)$ where n is any whole number.

Task 6

After every 7 days, Monday repeats. So after 14 days it's a Monday again, and again on the 21st and even 28th. Now after 29 days, the day is Tuesday, 30 days would be Wednesday and hence the 31st day would be Thursday.

Alternatively, $31 \equiv 3 \pmod{7}$, hence the problem is equivalent to asking what comes 3 days after Monday, which is Thursday.

We first list the number of days left to Christmas in each month

April - 29

May - 31

June - 30

July - 31

August - 31

September - 30

October - 31

November - 30

December - 25

$268 \equiv 2 \pmod{7}$ which means Friday ($1 \equiv \pmod{7}$) and then Saturday ($2 \equiv \pmod{7}$) from Thursday.

This year is a leap year hence it has $366 \text{ days} \equiv 2 \pmod{7}$. Thus the 1st Jan next year would be a Sunday.

Interestingly, we can also know when 1st Jan will be a Friday again.

Exactly 1 year later, we get Sunday. Now, this would be non-leap year with $365 \text{ days} \equiv 1 \pmod{7}$. Hence, 2 years later we get to Monday. This would repeat again for two more non-leap years. And so, 4 years later, 1st Jan is Wednesday. Now the fifth year would be 366

days $\equiv 2 \pmod{7}$, and moving two days from Wednesday gives Friday again! So we get 1st Jan as a Friday again after 5 years.

Today is 13 May, Thursday. My birthday is on 1st October. Each month has these many days:

May - 18

June - 30

July - 31

August - 31

September - 30

October - 1

141 days $\equiv 1 \pmod{7}$, 1 day from Thursday, which is Friday. And 1st October 2021 is a Friday indeed.