Euler's Totient Function

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Question 1

$\phi(15)$ is equal to all numbers less than 15 that are co-prime to it.

There are 8 numbers: 1, 2, 4, 7, 8, 11, 13, and 14. Thus $\phi(15) = 8$

Question 2

$\phi(p)$ is equal to all numbers less than p that are co-prime to it. We see that all numbers except p are in fact co-prime to it. Thus $\phi(p) = p-1$

As an example, $\phi(3) = 2$, $\phi(5) = 4$, $\phi(7) = 6$, $\phi(101) = 100$ where each of these are prime numbers.

Question 3

We saw that $\phi(p) = p - 1$. Returning back to the original definition of $\phi(n)$ which is equal to number of positive integers less than n which are co-prime to n. From another way, $\phi(n)$ is also equal to n - (positive integers not co-prime to n).

For $\phi(2^n)$, we can use this fact. $\phi(2^n) = 2^n$ - (positive integers not co-prime to 2^n). All positive integers that are not co-prime are $1 \times 2,\ 2 \times 2,\ 3 \times 2,\ 4 \times 2,\ \ldots\ 2^{n-1} \times 2$ which are a total of 2^{n-1} numbers.

$\phi(2^n) = 2^n - 2^{n-1} = 2^{n-1}\,(2 - 1) = 2^{n-1}$

Very similarly, $\phi(3^n) = 3^n - 3^{n-1} = 3^{n-1}\,(3 - 1) = 2 \times 3^{n-1}$

In general, $\phi(p^n) = p^n - p^{n-1} = p^{n-1}\,(p - 1)$

Question 4

$\phi(15) = 8$,

We observed and proved that $\phi(p) = p - 1$

$\phi(3) = 2$ since 3 is a prime and $\phi(5) = 4$ since 5 is a prime too.

And we see that $\phi(15) = 8 = \phi(3) \times \phi(5)$

To explore more, I took $\phi(3) \times \phi(3) = 4 \neq \phi(9)$

$\phi(2) \times \phi(4) = 1 \times 2 \neq \phi(8)$

So, this function isn't multiplicative for <u>any two</u> integers. I think it only works when the two integers are co-prime as well, according to my observations.

Question 5

$\phi(n)$ includes n which includes prime factors raised to powers. $n = p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \ldots p_m^{k_m}$

$\phi(n) = \phi(p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \ldots p_m^{k_m})$

Now, in Question 4 we observed that this function is multiplicative when integers are co-prime. And all pairs of integers $p_i^{k_i}$ would definitely be co-prime to each other.

$\phi(n) = \phi(p_1^{k_1}) \times \phi(p_2^{k_2}) \times \phi(p_3^{k_3}) \ldots \phi(p_m^{k_m})$

In Question 3, $\phi(p^n) = p^n - p^{n-1} = p^{n-1}\ (p - 1)$

$\phi(n) = p_1^{k_1-1}\ (p_1 - 1) \times p_2^{k_2-1}\ (p_2 - 1) \times p_3^{k_3-1}\ (p_3 - 1) \ldots p_m^{k_m-1}\ (p_m - 1)$

However, there's a clever step to instead take the entire prime power common.

$$\phi(n) = p_1^{k_1}\ \left(1 - \frac{1}{p_1}\right) \times p_2^{k_2}\ \left(1 - \frac{1}{p_2}\right) \times p_3^{k_3}\ \left(1 - \frac{1}{p_3}\right) \ldots p_m^{k_m}\ \left(1 - \frac{1}{p_m}\right)$$

$$\phi(n) = p_1^{k_1}\, p_2^{k_2}\, p_3^{k_3} \ldots p_m^{k_m} \left(1 - \frac{1}{p_1}\right) \times \left(1 - \frac{1}{p_2}\right) \times \left(1 - \frac{1}{p_3}\right) \ldots \left(1 - \frac{1}{p_m}\right)$$

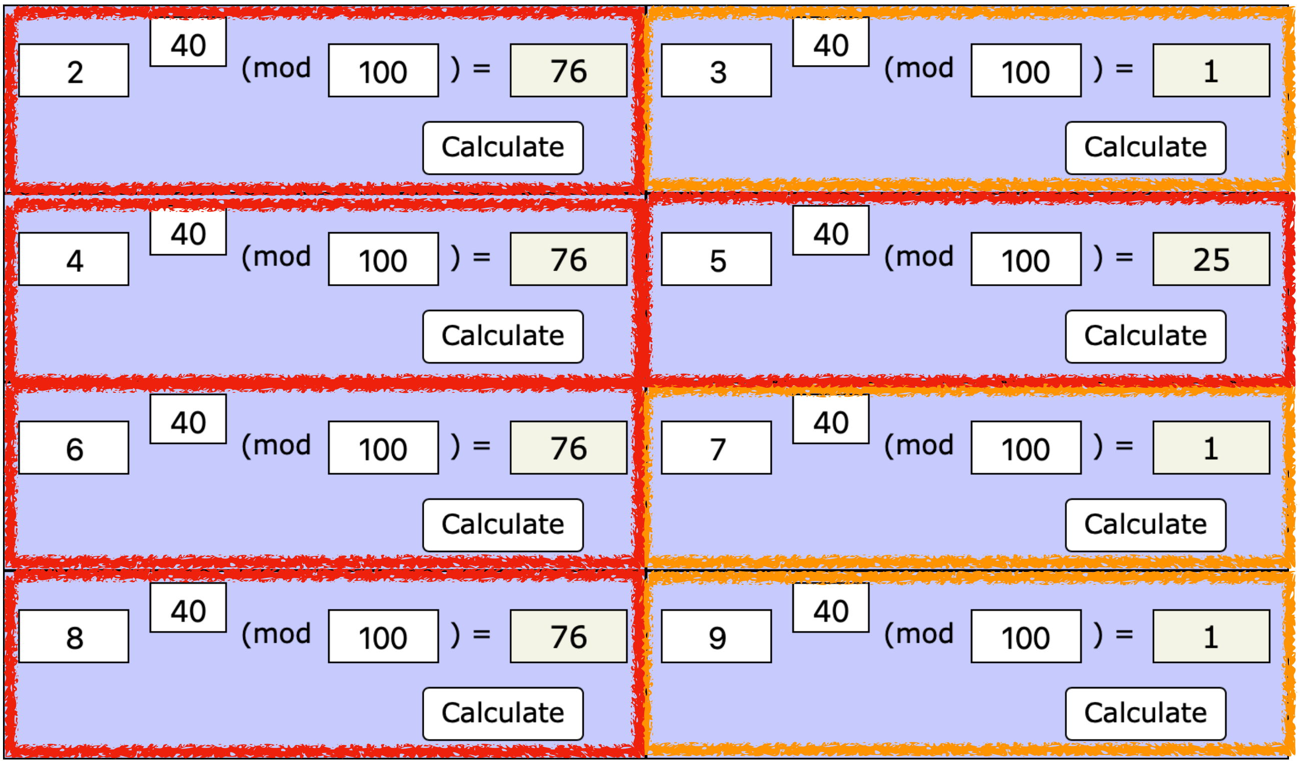
$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \times \left(1 - \frac{1}{p_2}\right) \times \left(1 - \frac{1}{p_3}\right) \ldots \left(1 - \frac{1}{p_m}\right)$$

Euler's Totient Function is always even!! Let n be $p^k \times m$

$$\begin{aligned} \phi(n) &= \phi(p^k \times m) \\ &= \phi(p^k) \times \phi(m) \\ &= p^{k-1}(p-1) \times \phi(m) \end{aligned}$$

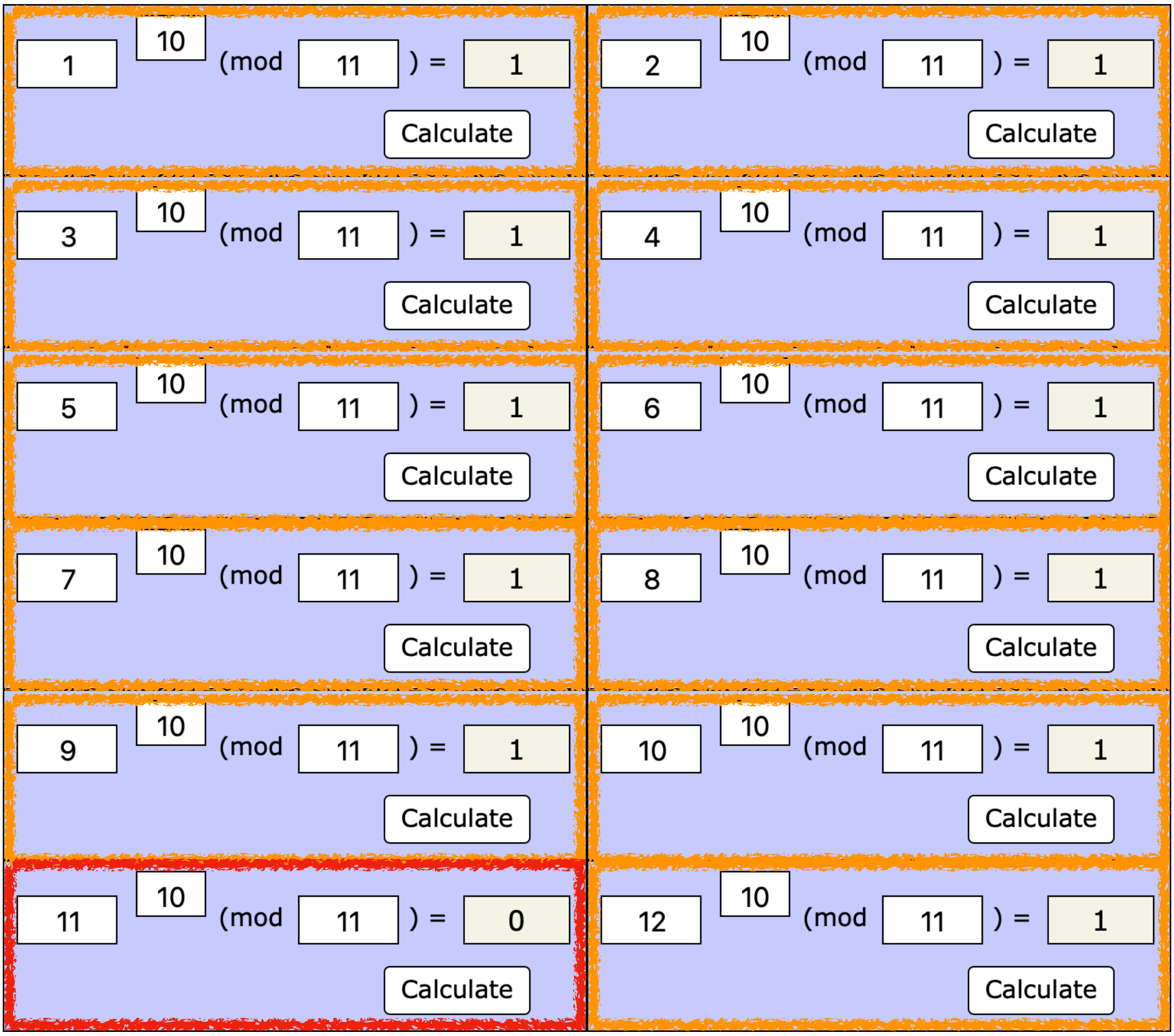
$\phi(n)$ includes the factor (p-1) which is always even.

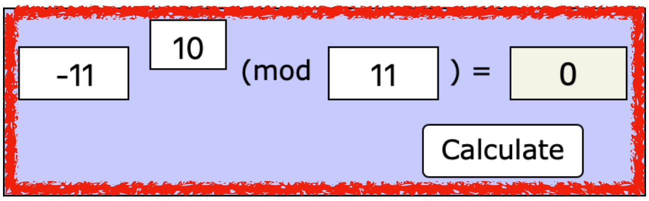
Extension:



I tried a couple of values into the modulo calculator. And it only worked when x and n were co-prime

For an input with modulo prime, all values for a seemed to work $a^{p-1} \equiv 1 \mod p$ except when a is a multiple of p.





This even works for negative integers!