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# Euler's Totient function

The function of $\phi(n)$ for a positive integer n is defined by,
$\Phi(n)$ = The number of positive integer [1,n] which are co-primes to n.

Two numbers are co-primes if the gcd of (a, b) is 1.

Question 1)

Show that $\phi(15) = 8$

$\varphi(15)$ $\{\underline{1}, \underline{2}, 3, \underline{4}, 5, 6, \underline{7}, \underline{8}, 9, 10, \underline{11}, 12, \underline{13}, \underline{14}, 15\}$

By adding all co-primes we get 8, $\therefore \varphi(15) = 8$

Question 2)

Investigate $\phi(P)$ where p is a prime number. Can you find a general expression for $\phi(P)$?

- $\varphi(5)$ $\{\underline{1}, \underline{2}, \underline{3}, \underline{4}, 5\}$ $\therefore \varphi(5) = 4$
- $\varphi(7)$ $\{\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, 7\}$ $\therefore \varphi(7) = 6$
- $\varphi(11)$ $\{\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, 11\}$ $\therefore \varphi(11) = 10$
- $\varphi(P)$ $\{\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \ldots\ldots, P\}$

When P is prime, none of the numbers smaller than P will share any factors with P, as they have to include P in their prime factorisation. Thus, all the numbers smaller than P are co-primes to P. Hence, it can be said that $\phi(P)$ will evaluate to P-1 coprimes.

$\therefore \phi(P) = P-1$

Question 3)

Investigate $\phi(2^n)$ where $n = 1, 2, 3, \ldots$ Can you find a general expression for $\phi(2^n)$? What about $\phi(3^n)$?
Can you find a general expression for $\phi(p^n)$ where $p$ is prime?

$\phi(2^n) \rightarrow$

- $\phi(2^1) = 1$
- $\phi(2^2) = 2$
- $\phi(2^3) = 4$

$\vdots$

- $\phi(2^7) = 64$

It can be seen that the expression for $\phi(2^n)$ is $2^{n-1}$

$\phi(3^n) \rightarrow$

- $\phi(3^1) = 2$
- $\phi(3^2) = 6$
- $\phi(3^3) = 18$

$\vdots$

- $\phi(3^7) = 1{,}458$

It can be seen that the expression for $\phi(3^n)$ is $3^n - 3^{n-1}$

Thus,

$\phi(p^n)$ $\{\underbrace{p, 2p, 3p, 4p, 5p,}_{\text{co-primes}} \ldots, p^n\}$

$\downarrow$

$p^{a-1}(p)$

Hence:

$$\phi(p^n) = p^n - p^{n-1}$$
$$= p^{n-1}(p-1), \text{ where } p \text{ is prime}$$

Question 4)

In Question 1 you showed that $\phi(15) = 8$. What are $\phi(3)$ and $\phi(5)$?
Is it true that $\phi(15) = \phi(3) \times \phi(5)$?
Under which conditions is $\phi(nm) = \phi(n) \times \phi(m)$ true?

$\phi(15) = 8$
$\phi(5) = 4$
$\phi(3) = 2$
$\therefore \phi(15) = \phi(5) \cdot \phi(3)$

Alternative example:

$\phi(1{,}333) = 1{,}260$
$\phi(31) = 30$
$\phi(43) = 42$
$\therefore \phi(1{,}333) = \phi(31) \cdot \phi(43)$

Counter example:

$\phi(54) = 18$
$\phi(9) = 6$
$\phi(6) = 2$
$\therefore \phi(54) \neq \phi(9) \cdot \phi(6)$
as $6 \times 2 \neq 18$

The Condition for $\phi(nm) = \phi(n) \cdot \phi(m)$ to be true is that both n and m have to be co-primes to each other. 2 numbers have the co-prime property if $\gcd(a, b) = 1$

Hence, It can be said that $\phi(nm) = \phi(n) \cdot \phi(m)$ if n and m are co-primes to each other.

Question 5)

Can you find a general expression for $\phi(n)$?

Considering the prime factorisation to be;
$n = a_1^{b1} \cdot a_2^{b2} \cdot a_3^{b3} \cdot a_n^{bn}$
we also know that $\phi(p^n) = p^{n-1}(p-1)$

So we will obtain,

$$(a_1^{b1} - a_1^{b1-1}) \cdot (a_2^{b2} - a_2^{b2-1}) \cdot \ldots \cdot (a_n^{bn} - a_n^{bn-1})$$

$$\Rightarrow a_1^{b1}\left(1 - \frac{1}{a_1}\right) \cdot a_2^{b2}\left(1 - \frac{1}{a_2}\right) \ldots a_n^{bn}\left(1 - \frac{1}{a_n}\right)$$

$$= \underbrace{a_1^{b1} a_2^{b2} \ldots a_n^{bn}}\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right)\ldots\left(1 - \frac{1}{a_n}\right)$$

$a_1^{b1} a_2^{b2} \ldots a_n^{bn}$ are the prime factorisation of $n$

Simplified:

$$\phi(n) = n\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right)\ldots\left(1 - \frac{1}{a_n}\right)$$

where $a_1, a_2, a_n$ are distinc prime factors of $n$

Examples:

- $\phi(50) = 20$
  $= 2 \times 5^2$
  $= 50\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$
  $= 50\left(\frac{1}{2}\right)\left(\frac{4}{5}\right) = \underline{20}$

- $\phi(2021) = 1932$
  $= 43 \times 47$
  $= 2021\left(1 - \frac{1}{43}\right)\left(1 - \frac{1}{47}\right)$
  $= 2021\left(\frac{42}{43}\right)\left(\frac{46}{47}\right) = \underline{1932}$