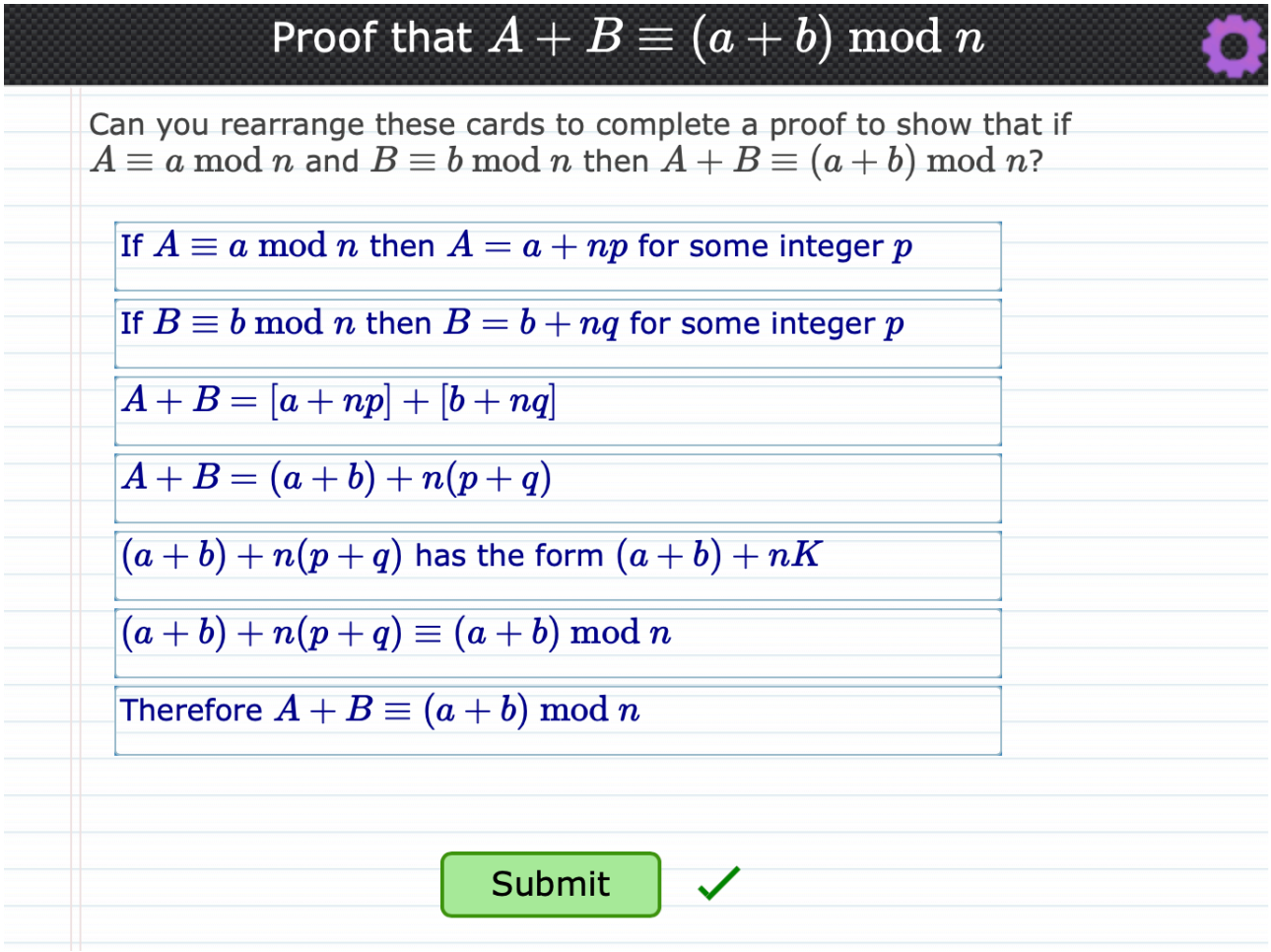
More Adventures with Modular Arithmetic

Mahdi Raza Khunt
Mahatma Gandhi International School



The proofs for $A + B \equiv (a + b) \mod n$ and $AB \equiv ab \mod n$ match.

Equations with modulo arithmetic

$$\begin{aligned} 3x &\equiv 1 \mod 7 \\ 3x &\equiv 15 \mod 7 \\ x &\equiv 5 \mod 7 \end{aligned} \qquad \begin{aligned} 3x &\equiv 2 \mod 7 \\ 3x &\equiv 9 \mod 7 \\ x &\equiv 3 \mod 7 \end{aligned} \qquad \begin{aligned} 3x &\equiv 3 \mod 7 \\ 3x &\equiv 3 \mod 7 \\ x &\equiv 1 \mod 7 \end{aligned} \qquad \begin{aligned} 3x &\equiv 4 \mod 7 \\ 3x &\equiv 18 \mod 7 \\ x &\equiv 6 \mod 7 \end{aligned}$$

$$\begin{aligned} 3x &\equiv 5 \mod 7 \\ 3x &\equiv 12 \mod 7 \\ x &\equiv 4 \mod 7 \end{aligned} \qquad \begin{aligned} 3x &\equiv 6 \mod 7 \\ 3x &\equiv 6 \mod 7 \\ x &\equiv 2 \mod 7 \end{aligned} \qquad \begin{aligned} 3x &\equiv 0 \mod 7 \\ 3x &\equiv 0 \mod 7 \\ x &\equiv 0 \mod 7 \end{aligned}$$

$6x \equiv 1 \mod 7$
$6x \equiv 36 \mod 7$
$x \equiv 6 \mod 7$

$6x \equiv 2 \mod 7$
$6x \equiv 30 \mod 7$
$x \equiv 5 \mod 7$

$6x \equiv 3 \mod 7$
$6x \equiv 24 \mod 7$
$x \equiv 4 \mod 7$

$6x \equiv 4 \mod 7$
$6x \equiv 18 \mod 7$
$x \equiv 3 \mod 7$

$6x \equiv 5 \mod 7$
$6x \equiv 12 \mod 7$
$x \equiv 2 \mod 7$

$6x \equiv 6 \mod 7$
$6x \equiv 6 \mod 7$
$x \equiv 1 \mod 7$

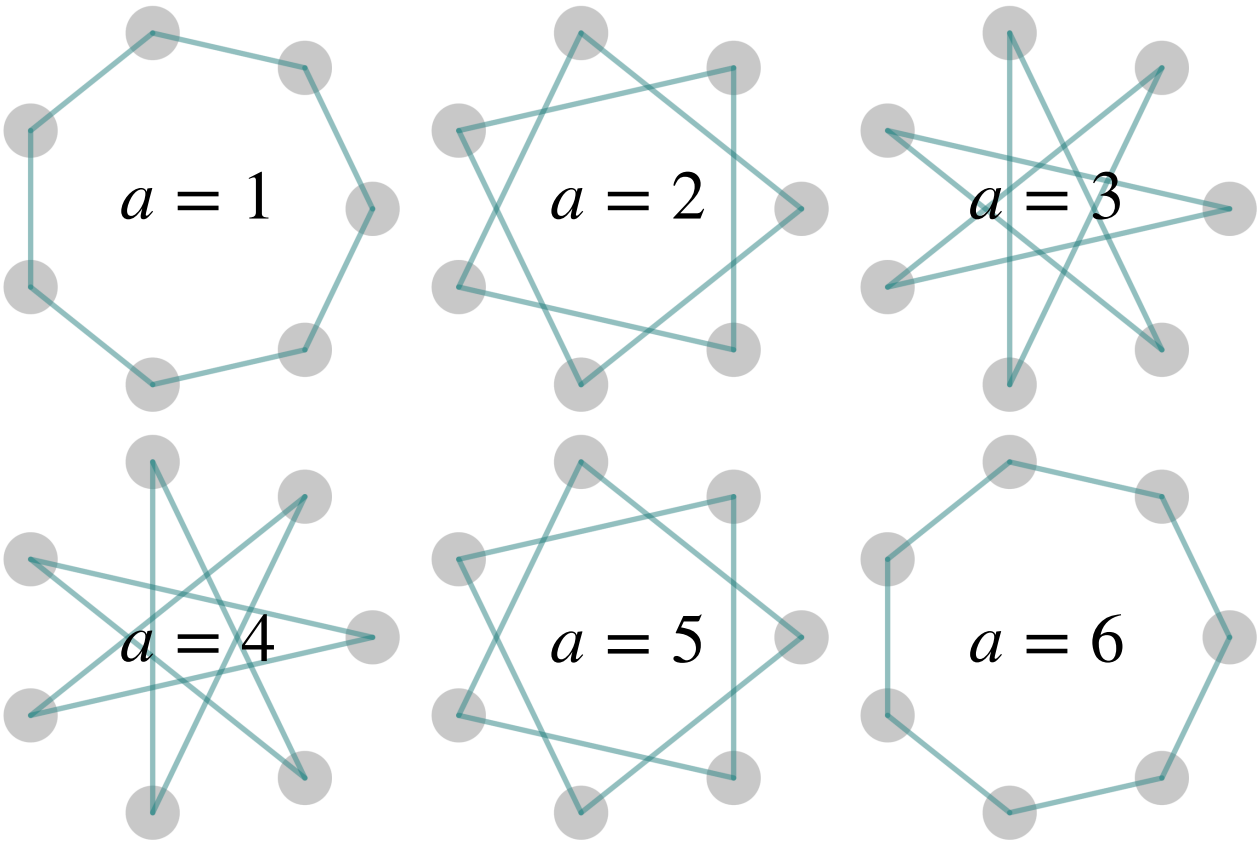
$6x \equiv 0 \mod 7$
$6x \equiv 0 \mod 7$
$x \equiv 0 \mod 7$

Since 7 is prime, for every value of a and b, there is a unique solution for x in the expression: $a\,x \equiv b \mod 7$

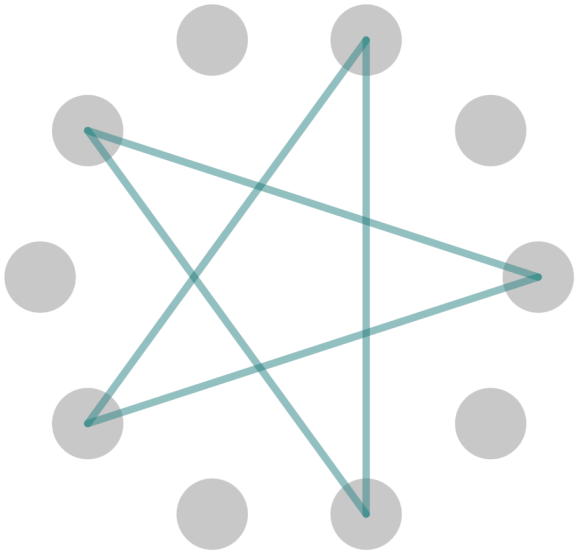
For a = 3 and b = 1,2,3,4,5,6,0

And for a = 6 and b = 1,2,3,4,5,6,0

We have seen that it is true. It is also true for a = 1, 2, 4, and 5



Another observation I see is that if for some a there exists a unique solution for all b, then for step size $(n - a)$, there must exist a unique solution for all b as well.



In this case of $4x \equiv b \mod 10$, we see that there are only 5 values of b that are namely: 2, 4, 6, 8 and 0

To cover all steps, as seen in the Star Problem as well, we require to choose the step size as a number that is co-prime to the number of points chosen.

So, when a and n are co-prime, we get a solution for all values of b from $0,1,2\ldots n-1$

To work out the number of solutions for this problem, we consider $\phi(n)$ which is the Euler's Totient Function. It will help us to determine the number of values of a that exist and have solutions for all values of b from $0,1,2\ldots n-1$