Tetra Slice

We may represent the points A, B, C, and D as position vectors from some fixed origin O such that:

$\overrightarrow{OA} = a$
$\overrightarrow{OB} = b$
$\overrightarrow{OC} = c$
$\overrightarrow{OD} = d$

Then we look to express each side of quadrilateral PQRS using these vectors so starting with side PS:

$$\overrightarrow{PS} = \overrightarrow{PA} + \overrightarrow{AS} = \tfrac{1}{2}\overrightarrow{BA} + \tfrac{1}{2}\overrightarrow{AC} = \frac{a-b}{2} + \frac{c-a}{2} = \frac{c-b}{2}$$

Note that we can find $\overrightarrow{BA}$ and $\overrightarrow{AC}$ by going from the first point to the origin then to the second point e.g.

$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -b + a = a - b$$

Now we need to find an expression for every other side of quadrilateral PQRS so:

$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BD}$ and using similar arguments as in the note we get

$$\tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BD} = \frac{b-a}{2} + \frac{d-b}{2} = \frac{d-a}{2}$$

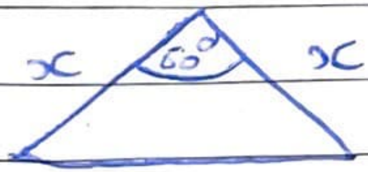
$$\overrightarrow{QR} = \overrightarrow{QD} + \overrightarrow{DR} = \tfrac{1}{2}\overrightarrow{BD} + \tfrac{1}{2}\overrightarrow{DC} = \frac{d-b}{2} + \frac{c-d}{2} = \frac{c-b}{2}$$

$$\overrightarrow{SR} = \overrightarrow{SC} + \overrightarrow{CR} = \tfrac{1}{2}\overrightarrow{AC} + \tfrac{1}{2}\overrightarrow{CD} = \frac{c-a}{2} + \frac{d-c}{2} = \frac{d-a}{2}$$

So we can see $\overrightarrow{PS} = \overrightarrow{QR}$ and $\overrightarrow{PQ} = \overrightarrow{SR}$

So PS and QR are parallel and also PQ and SR are parallel so we have two pairs of parallel thus quadrilateral PQRS is a parallelogram. □

Extension: If ABCD is regular we know all the angles in the triangles are 60° so each length of PQRS can be found using the same triangle:



where the bottom side is the side of PQRS. Since this triangle is the same for all sides of PQRS it means each side of PQRS is equal so it is a square.