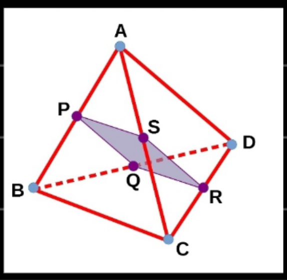
Q. Prove the midpoints of a tetrahedron forms a parallelogram.



Let the position vectors of A, B, C and D be $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ respectively.

We can then find the vectors corresponding to the sides of the parallelogram.

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PA} + \overrightarrow{AS} \\ &= \tfrac{1}{2}\overrightarrow{BA} + \tfrac{1}{2}\overrightarrow{AC} \\ &= \tfrac{1}{2}(\underline{a}-\underline{b}) + \tfrac{1}{2}(\underline{c}-\underline{a}) \\ &= \cancel{\tfrac{1}{2}\underline{a}} - \tfrac{1}{2}\underline{b} + \tfrac{1}{2}\underline{c} - \cancel{\tfrac{1}{2}\underline{a}} \\ &\boxed{= \tfrac{1}{2}(\underline{c}-\underline{b})}\end{aligned}$$

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QD} + \overrightarrow{DR} \\ &= \tfrac{1}{2}\overrightarrow{BD} + \tfrac{1}{2}\overrightarrow{DC} \\ &= \tfrac{1}{2}(\underline{d}-\underline{b}) + \tfrac{1}{2}(\underline{c}-\underline{d}) \\ &= \cancel{\tfrac{1}{2}\underline{d}} - \tfrac{1}{2}\underline{b} + \tfrac{1}{2}\underline{c} - \cancel{\tfrac{1}{2}\underline{d}} \\ &\boxed{= \tfrac{1}{2}(\underline{c}-\underline{b})}\end{aligned}$$

We can see that $\overrightarrow{PS} = \overrightarrow{QR}$ and is therefore a set of parallel sides of equal length.

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ} \\ &= \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BD} \\ &= \tfrac{1}{2}(\underline{b}-\underline{a}) + \tfrac{1}{2}(\underline{d}-\underline{b}) \\ &= \cancel{\tfrac{1}{2}\underline{b}} - \tfrac{1}{2}\underline{a} + \tfrac{1}{2}\underline{d} - \cancel{\tfrac{1}{2}\underline{b}} \\ &\boxed{= \tfrac{1}{2}(\underline{d}-\underline{a})}\end{aligned}$$

$$\begin{aligned}\overrightarrow{SR} &= \overrightarrow{SC} + \overrightarrow{CR} \\ &= \tfrac{1}{2}\overrightarrow{AC} + \tfrac{1}{2}\overrightarrow{CD} \\ &= \tfrac{1}{2}(\underline{c}-\underline{a}) + \tfrac{1}{2}(\underline{d}-\underline{c}) \\ &= \cancel{\tfrac{1}{2}\underline{c}} - \tfrac{1}{2}\underline{a} + \tfrac{1}{2}\underline{d} - \cancel{\tfrac{1}{2}\underline{c}} \\ &\boxed{= \tfrac{1}{2}(\underline{d}-\underline{a})}\end{aligned}$$

We can see here that $\overrightarrow{PQ} = \overrightarrow{SR}$, and therefore are two sides that are parallel to each other with equal length.

Therefore, we have proven that PQRS is a parallelogram.

Ext. If ABCD is a regular tetrahedron, what else can you say about PQRS?

If ABCD is a regular tetrahedron, that means the lengths of its sides are equal.

two sides of the tetrahedron

$\Rightarrow |\overrightarrow{BC}| = |\overrightarrow{AD}|$

the lengths of the sides of the parallelogram

$\Rightarrow |\underline{c}-\underline{b}| = |\underline{d}-\underline{a}|$ and $\tfrac{1}{2}|\underline{c}-\underline{b}| = \tfrac{1}{2}|\underline{d}-\underline{a}|$

$\therefore$ the sides of the parallelogram are all equal and PQRS is a <u>square</u>.