1 Let k be an integer.
Then, $2(k)$ is an even number and any number in this form is even.
Moreover, $2(k)+1$ is an odd number and any number in this form is odd.

We conclude, for ease of later proofs:

1) Any even number multiplied by another even number is equal to an even number.

$$(2k)(2n) = 4kn$$
$$= 2(2kn) \rightarrow \text{where } 2kn \text{ is some number.}$$
$\hookrightarrow$ ($2(2kn)$ is in the form $2k$)

2) Any odd number multiplied by another odd number is equal to an odd number:

$$(2k+1)(2n+1) = 4kn + 2k + 2n + 1$$
$$= 2(2kn + k + n) + 1 \rightarrow \text{where } 2kn+k+n \text{ is some number}$$
$\hookrightarrow$ ($2(2kn+k+n)+1$ is in the form $2k+1$)

3) Any even number multiplied by an odd number is equal to an even number:

$$(2k)(2n+1) = 2kn + 2k$$
$$= 2(kn + k) \rightarrow \text{where } kn+k \text{ is some number}$$
$\hookrightarrow$ ($2(kn+k)$ is in the form $2k$

4) Any even number plus an even number is even:
$$2k+2n = 2(k+n)$$

5) Any odd number plus an odd number is ~~odd~~ even
$\cancel{2k+1 \quad 2n+1 = 2(k+n)+1}$
$$2k+1 + 2n+1 = 2(k+n+1)$$

6) Any odd number plus an even number is odd:
$$2k+1 + 2n = 2(k+n)+1$$

(c)

Therefore, an odd square can only be obtained by multiplying two odd numbers (by proof 2)

∴ Since a square number occurs when a number is multiplied by itself, we adopt that:

$(2K+1)^2$ is an odd square

↳ $4K^2+4K+1$

~~As a result, dividing the result by 8 gives:~~

$4K^2+4K+1 = 4(K^2+K) + 1$

To conclude a remainder 1 is produced, we just have to prove that $4K^2+4K$ is divisible by 8 since this will leave the remainder of 1.

$$\frac{4(K^2+K)}{8} = \frac{K^2+K}{2}$$

↳ Using proofs 1, 2, 4, and 5:

Case 1

If K is even, then K^2 is also even (by proof 1)
Therefore, by proof 4, K^2+K is even so can be written in the format $2n$ where n is some number.
As a result it can be divisible by 2 ~~and leaves a remainder of 1 (in total, ie~~

Case 2

If K is odd, then K^2 is odd (by proof 2). Then K^2+K is even, by proof 5. This means it can be written in the form $2n$ where n is some number. This means K^2+K is divisible by 2.

~~1 Let any even number be~~

Both cases suggest that $4(K^2+K)$ is divisible by 8 therefore, a remainder of 1 is left behind when $4(K^2+K)+1$ is divided by 8.

Therefore, any ODD SQUARE number divided by 8 leaves remainder 1.

2 By proof 1, an even square can only be produced by squaring an even number. Therefore:

$(2K)^2$ is an even square

$(2K)^2 = 4K^2$

Dividing $4K^2$ by 8 gives:

$*$ $\frac{4K^2}{8} = \frac{K^2}{2}$

By proof 1 and 2:

Case 1:

If K is even, then K^2 is even. Therefore if K is even, no remainder is left i.e. a remainder of 0. HOWEVER, we divided the expression by 4 in step $*$. Therefore, we must multiply the remainder by 4. $4 \times 0 = \underline{0}$

Case 2:

If K is odd, then K^2 is odd. Therefore, if K is odd, a remainder of 1 is left.

HOWEVER, we divided the ~~result~~ expression by 4 in step $*$. we must multiply the remainder by 4 to get remainder $\underline{4}$. $(1 \times 4 = 4)$

④

③ We let three numbers (even or odd) be: a, b and c. NOTE, a, b and c are all INTEGERS.

Then, their squares are a^2, b^2, c^2

The sum of the squares are $a^2+b^2+c^2$.

To prove that the sum of the squares CAN NEVER EQUAL $8n+7$ we use a PROOF by CONTRADICTION

Let
$$a^2+b^2+c^2 = 8n+7.$$

We have 4 combinations of whether each number is even or odd: Let even = E Let odd = O

E + E + E
E + E + O
E + O + O
O + O + O

This is made easy since the square of an even number is even and the square of an odd number is odd (by proof 1 and 2).

Therefore:
<u>Case 1</u> -> E + E + E
If all three numbers a, b, c are even, all the squares are even (proof number 1).

$$a^2 = (2d)^2$$
$$b^2 = (2e)^2$$
$$c^2 = (2f)^2$$

1 ~~Let k be any number~~

Then

$$(2d)^2 + (2e)^2 + (2f)^2 = 8n+7$$
$$4d^2 + 4e^2 + 4f^2 = 8n+7$$
$$4(d^2 + e^2 + f^2) = 8n+7$$
$$2(2(d^2+e^2+f^2) = 8n+7$$
$$2(2(d^2+e^2+f^2) = 8n+6+1$$
$$2(2(d^2+e^2+f^2) = 2(4n+3)+1)$$

This is a CONTRADICTION since we are claiming that the even number: $2(2(d^2+e^2+f^2)$ is equal to the odd number $2(4n+3)+1$.

Therefore, CASE 1 is a contradiction.

Case 2: $\rightarrow$ E+E+O

Again using same notation:

$$(2d)^2 + (2e)^2 + (2f+1)^2 = 8n+7$$
$$4d^2 + 4e^2 + 4f^2 + 4f + 1 = 8n+7$$
$$4(d^2 + e^2 + f^2 + f) = 8n+6$$
$$2(2(d^2+e^2+f^2+f) = 2(4n+3)$$

For this claim to be true, we COMPARE COEFFICIENTS

so

$$2(2(d^2+e^2+f^2+f)) = 2(4n+3)$$
$$2(d^2+e^2+f^2+f) = 4n+3$$
$$2(d^2+e^2+f^2+f) = 2(2n+1)+1$$

HOWEVER This is a CONTRADICTION AS an even number CANNOT EQUAL an odd number.

And $2(d^2+e^2+f^2+f)$ is even whilst $2(2n+1)+1$ is odd.

Therefore, Case 2 is a contradiction.

Case 3 → E + O + O

Using the same notation:

$$(2d)^2 + (2e+1)^2 + (2f+1)^2 = 8n+7$$
$$4d^2 + 4e^2+4e+1 + 4f^2+4f+1 = 8n+7$$
$$4(d^2+e^2+e+f^2+f) = 8n+5$$
$$2(2(d^2+e^2+e+f^2+f) = 2(4n+2)+1$$

~~By comparing~~

ONCE AGAIN WE REACH A CONTRADICTION AS EVEN NUMBER $2(2(d^2+e^2+e+f^2+f)$ cannot equal odd number $2(4n+2)+1$.

Therefore Case 3 is a contradiction.

Case 4 → O + O + O

Using the same notation:

$$(2d+1)^2 + (2e+1)^2 + (2f+1)^2 = 8n+7$$
$$4d^2+4d+1 + 4e^2+4e+1 + 4f^2+4f+1 = 8n+7$$
$$4(d^2+d+e^2+e+f^2+f) = 8n+4$$
$$4(d^2+d+e^2+e+f^2+f) = 4(2n+1)$$

By comparing coefficients:

$$d^2+d+e^2+e+f^2+f = 2n+1$$

Now, $d^2+d+e^2+e+f^2+f$ MUST BE ODD FOR THE STATEMENT TO NOT BE CONTRADICTED.

However:

There are two cases:

If any of the numbers ~~is~~ are even
If any of the numbers ~~is~~ are odd.

~~If~~ Case 1

By proof 1, if K is even so is K^2
By proof 4 the sum is even.

Case 2

By proof 2 if K is odd, so is K^2
By proof 5 the sum is even.

This presents that if either d, e or f are either even or odd, the sum of the number and its square is even.

Therefore we have $(d^2+d)+(e^2+e)+(f^2+f)$
EVEN + EVEN + EVEN

By proof 4, the sum of two even numbers are even.

- Using proof 4 again, the sum of two even numbers are even.

Therefore the expression:
$d^2+d+e^2+e+f^2+f$ is even

This is A CONTRADICTION AS WE SUPPOSED ~~that it~~ EVEN = ODD.

8

Therefore, all 4 possible cases have been disapproved and CONTRADICTED ~~therefore~~, the expression $8n+7$

CAN NEVER be expressed as the sum of three squares.