Square Remainders

Mahe

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Problem

Show that every odd square leaves remainder 1 when divided by 8, and that every even square leaves remainder 0 or 4.

Deduce that a number of the form $8n + 7$, where n is a positive integer, cannot be expressed as a sum of three squares.

STEP Mathematics III, Specimen paper, Q10(i). Question reproduced by kind permission of Cambridge Assessment Group Archives.

My Solution

I started off by writing a list of the first 10 squares numbers, applying modular arithmetic of 8.

$$1^2 = 1(mod 8) = 1$$

$$2^2 = 4(mod 8) = 4$$

$$3^2 = 9(mod 8) = 1$$

$$4^2 = 16(mod 8) = 0$$

$$5^2 = 25(mod 8) = 1$$

$$6^2 = 36(mod 8) = 0$$

$$7^2 = 49(mod 8) = 1$$

$$8^2 = 64(mod 8) = 0$$

$$9^2 = 81(mod 8) = 1$$

$$10^2 = 100(mod 8) = 4$$

I noticed that an odd square number is produced by the product of an odd number. This is also true for an even square number.

An odd number is produced by the expression $2n + 1$, where n starts with 0, producing every positive odd number: $1, 3, 5, 7, 9, 11, 13, 15, 17, 19, ...$ I initially tried to form a odd square number using $2n + 1$.

$$(2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 4n(2n + 1) + 1$$

Despite having a remainder of 1, it is not a multiple of 8

Using $2n+1$ won't work, even though it represents every odd number. This means that I had to find another expression that represents every odd number that gives remainder 1 when divided by 8.

In order to be divisible by 8 the odd number squared should give a number that is a factor of 8, therefore a reasonable expression is using the next even number $4n+1$. However, starting with $n=0$, this expression provides with the sequence $1, 5, 9, 13, 17, ...$ and this seems to represents half of all the positive odd numbers.
Therefore the other sequence of odd numbers should be starting with 3, which means the expression is $4n+3$, staring with $n=0$. The square of these expression gives:

$$(4n+1)(4n+1)$$
$$16n^2+8n+1$$
$$8n(2n+1)+1$$

$$(4n+3)(4n+3)$$
$$16n^2+24n+9$$
$$16n^2+24n+8+1$$
$$8(2n^2+3n+1)+1$$
$$8(2n+1)(n+1)+1$$

Both $8n(2n+1)+1$ and $8(2n+1)(n+1)+1$ have a factor of 8, which means it is divisible by 8, and have a 1 on the outside, which represents the remainder. As $4n+1$ and $4n+3$ leaves remainder 1 when divided by 8, therefore it is true for every odd square.

For even numbers, I took the same approach. I saw that even though $2n$ gives every even number, this expression squared $4n^2$ is not a factor of 8. Therefore it is best using 2 expression, each representing half of the sequences, which are $4n$ $(0, 4, 8, 12, 16, 20, ..)$ and $4n + 2$ $(2, 6, 10, 14, 28, ..)$. Squaring these expressions will give you:

$$(4n)(4n) = 16n^2 = 8n(2n)$$

$$(4n + 2)(4n + 2) = 16n^2 + 16n + 4 = 8n(2n + 2) + 4$$

Both expressions can be divisible by 8, as the both have a factor of 8, however $8n(2n)$ has no remainder and $8n(2n+2)+4$ has a remainder of 4, and therefore since $4n$ and $4n + 2$ represent every even number, the square of these numbers divided by 8 leaves with remainder 4 or 0.

For the second part of the problem, I immediately wrote this expression.

$$x^2 + y^2 + z^2 = 8n + 7$$

where x, y and z can be any positive integer.

However, I realised that these numbers can either be even or odd (using the expressions above since the sum is a factor of 8 with remainder 7), therefore I wrote all the possible combinations of having 3 numbers, either even or odd

0 odd numbers:
even + even + even

1 odd numbers:
even + even + odd
even + odd + even
odd + even + even

2 odd numbers:
even + odd + odd
odd + even + odd
odd + odd + even

3 odd numbers:
odd + odd + odd

Since addition is commutative, I added the remainder of the even and odd number together.

- For 0 odd numbers: remainder for even number can be either 0 or 4, and therefore all possible remainders are 0, 4, 8, 12.
  However if remainder is bigger than 8, perform modular arithmetic of 8. This means that the actual possible remainders are 0 and 4.
- For 1 odd number: all possible remainders are 1, 5, 9. Therefore remainders are 1 and 5.
- For 2 odd number: all possible remainders are 2 and 6.
- For 2 odd number: only possible remainder is 3.

By looking at all the combinations, all possible remainders are 0, 1, 2, 3, 4, 5 and 6. This indicates that the sum three squares cannot have a remainder of 7.

My Written Solution