1 Let K be an integer.
Then, $2(K)$ is an even number and any number in this form is even.
Moreover, $2(K)+1$ is an odd number and any number in this form is odd.

We conclude, for ease of later proofs:

1) Any even number multiplied by another even number is equal to an even number.

$$(2K)(2n) = 4Kn$$
$$= 2(2Kn) \rightarrow \text{where } 2Kn \text{ is some number.}$$
$\hookrightarrow$ ($2(2Kn)$ is in the form $2K$)

2) Any odd number multiplied by another odd number is equal to an odd number:

$$(2K+1)(2n+1) = 4Kn + 2K + 2n + 1$$
$$= 2(2Kn + K + n) + 1 \rightarrow \text{where } 2Kn+K+n \text{ is some number}$$
$\hookrightarrow$ ($2(2Kn+K+n)+1$ is in the form $2K+1$)

3) Any even number multiplied by an odd number is equal to an even number:

$$(2K)(2n+1) = 2Kn + 2K$$
$$= 2(Kn + K) \rightarrow \text{where } Kn + K \text{ is some number}$$
$\hookrightarrow$ ($2(Kn+K)$ is in the form $2K$

4) Any even number plus an even number is even:
$$2K + 2n = 2(K+n)$$

5) Any odd number plus an odd number is ~~odd~~ even:
$\cancel{2K+1 \; 2n+1 = 2(K+n)+1}$
$$2K+1 + 2n+1 = 2(K+n+1)$$

6) Any odd number plus an even number is odd:
$$2K+1 + 2n = 2(K+n)+1$$

(c)

Therefore, an odd square can only be obtained by multiplying two odd numbers (by proof 2)

∴ Since a square number occurs when a number is multiplied by itself, we adopt that:

$(2K+1)^2$ is an odd square
↳ $4K^2+4K+1$

~~As a result, dividing the result by 8 gives:~~

$4K^2+4K+1 = 4(K^2+K)+1$
To conclude a remainder 1 is produced, we just have to prove that $4K^2+4K$ is divisible by 8 since this will leave the remainder of 1.

$$\frac{4(K^2+K)}{8} = \frac{K^2+K}{2}$$

↳ Using proofs 1, 2, 4, and 5:

Case 1
If K is even, then K^2 is also even (by proof 1)
Therefore, by proof 4, K^2+K is even so can be written in the format $2n$ where n is some number.
As a result it can be divisible by 2 ~~and leaves a remainder of 1 (in total, ie~~

Case 2

If K is odd, then K^2 is odd (by proof 2). Then K^2+K is even, by proof 5. This means it can be written in the form $2n$ where n is some number. This means K^2+K is divisible by 2.

1 ~~Let any even number be~~

Both cases suggest that $4(K^2+K)$ is divisible by 8 therefore, a remainder of 1 is left behind when $4(K^2+K)+1$ is divided by 8.

Therefore, any ODD SQUARE when divided by 8 leaves remainder 1.

2 By proof 1, an even square can only be produced by squaring an even number. Therefore:

$(2K)^2$ is an even square

$(2K)^2 = 4K^2$

Dividing $4K^2$ by 8 gives:

$*$ $\frac{4K^2}{8} = \frac{K^2}{2}$

By proof 1 and 2:

Case 1:
If K is even, then K^2 is even. Therefore if K is even, no remainder is left ie a remainder of 0. HOWEVER, we divided the expression by 4 in step $*$. Therefore, we must multiply the remainder by 4. $4 \times 0 = \underline{\underline{0}}$

Case 2:
If K is odd, then K^2 is odd. Therefore, if K is odd, a remainder of 1 is left.
HOWEVER, we divided the ~~result~~ expression by 4 in step $*$. we must multiply the remainder by 4 to get remainder $\underline{4}$ $(1 \times 4 = 4)$

3 We let three numbers (even or odd) be a, b and c. NOTE, a, b and c are all INTEGERS.

Then, their squares are a^2, b^2, c^2

The sum of the squares are $a^2+b^2+c^2$.

To prove that the sum of the squares CAN NEVER EQUAL $8n+7$ we use a PROOF by CONTRADICTION

Let
$a^2+b^2+c^2 = 8n+7$.

We have 4 combinations of whether each number is even or odd: Let even = E Let odd = O

E + E + E
E + E + O
E + O + O
O + O + O

This is made easy since the square of an even number is even and the square of an odd number is odd (by proof 1 and 2).

Therefore:
<u>Case 1</u> → E + E + E
If all three numbers a, b, c are even, all the squares are even (proof number 1).

$a^2 = (2d)^2$
$b^2 = (2e)^2$
$c^2 = (2f)^2$

1 ~~Let k be any number~~

Then

$(2d)^2 + (2e)^2 + (2f)^2 = 8n+7$

$4d^2 + 4e^2 + 4f^2 = 8n+7$

$4(d^2+e^2+f^2) = 8n+7$

$2(2(d^2+e^2+f^2)) = 8n+7$

$2(2(d^2+e^2+f^2)) = 8n+6+1$

$2(2(d^2+e^2+f^2)) = 2(4n+3)+1$

This is a CONTRADICTION since we are claiming that the even number: $2(2(d^2+e^2+f^2))$ is equal to the odd number $2(4n+3)+1$.

Therefore, CASE 1 is a contradiction.

Case 2: $\rightarrow$ E+E+O

Again using same notation:

$(2d)^2 + (2e)^2 + (2f+1)^2 = 8n+7$

$4d^2 + 4e^2 + 4f^2 + 4f + 1 = 8n+7$

$4(d^2+e^2+f^2+f) = 8n+6$

$2(2(d^2+e^2+f^2+f)) = 2(4n+3)$

For this claim to be true, we COMPARE COEFFICIENTS

so

$2(2(d^2+e^2+f^2+f)) = 2(4n+3)$

$2(d^2+e^2+f^2+f) = 4n+3$

$2(d^2+e^2+f^2+f) = 2(2n+1)+1$

HOWEVER This is a CONTRADICTION AS an even number CANNOT EQUAL an odd number.

And $2(d^2+e^2+f^2+f)$ is even whilst $2(2n+1)+1$ is odd.

Therefore, Case 2 is a contradiction.

Case 3 → E + O + O
Using the same notation:

$$(2d)^2 + (2e+1)^2 + (2f+1)^2 = 8n+7$$
$$4d^2 + 4e^2+4e+1 + 4f^2+4f+1 = 8n+7$$
$$4(d^2+e^2+e+f^2+f) = 8n+5$$
$$2(2(d^2+e^2+e+f^2+f) = 2(4n+2)+1$$

~~By comparing~~
ONCE AGAIN WE REACH A CONTRADICTION AS EVEN NUMBER $2(2(d^2+e^2+e+f^2+f)$ cannot equal odd number $2(4n+2)+1$.

Therefore Case 3 is a contradiction.

Case 4 → O+O+O
Using the same notation:

$$(2d+1)^2 + (2e+1)^2 + (2f+1)^2 = 8n+7$$
$$4d^2+4d+1 + 4e^2+4e+1 + 4f^2+4f+1 = 8n+7$$
$$4(d^2+d+e^2+e+f^2+f) = 8n+4$$
$$4(d^2+d+e^2+e+f^2+f) = 4(2n+1)$$

By comparing coefficients:

$$d^2+d+e^2+e+f^2+f = 2n+1$$

Now, $d^2+d+e^2+e+f^2+f$ MUST BE ODD FOR THE STATEMENT TO NOT BE CONTRADICTED.

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Therefore, all 4 possible cases have been disapproved and CONTRADICTED

~~therefore~~, the expression $8n+7$

CAN NEVER be expressed as the sum of three squares.