Proper Factors (NRICH Solution)

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1 Abstract

This document presents a solution to the "Proper Factors" problem published by the University of Cambridge's NRICH website. Methods used throughout this proof include the application of common number theory principles alongside basic algebra (e.g., finding the number of solutions to a specific equation.) I hope my explanations will help budding mathematicians better understand the intricacies behind solving this problem!

2 Solution (Part I)

For the first part of the problem, our aim is to show that,

$$3^2 \times 5^3 \tag{1}$$

has exactly 10 proper factors (i.e., all factors of a value except for the value itself.) So, multiplying the exponents gives us,

$$3 \times 2 = 6 \tag{2}$$

which represents how many combinations of a and b there are in the form $3^a \times 5^b$ as factors of $3^2 \times 5^3$, where $a \geq 1$ and $b \geq 1$. The value calculated in (2) includes $3^2 \times 5^3$ itself, so the number of proper factors using both 3 and 5 must be,

$$6 - 1 = 5 \tag{3}$$

We then account for factors in the form of 3^a and 5^b, whereby the total number of these values equates to,

$$a + b \tag{4}$$

since there are a factors from 3^a and b factors from 5^b. For $a = 2$ and $b = 3$, the total number of additional proper factors must be,

$$3 + 2 = 5 \tag{5}$$

Combining (3) and (5) gives us the total number of proper factors for $3^2 \times 5^3$, which equals to,

$$5 + 5 = 10 \tag{6}$$

This proves that $3^2 \times 5^3$ has exactly 10 proper factors. (Q.E.D.)

3 Solution (Part II)

For the second part of the problem, our aim is to find how many other integers of the form $3^m \times 5^n$ exist such that the total number of proper factors is 10. This value was determined using,

$$3 \times 2 + 3 + 2 - 1 \tag{7}$$

$$\Rightarrow 6 + 5 - 1 \tag{8}$$

$$\Rightarrow 10 \tag{9}$$

where $m = 3$ and $n = 2$. Therefore, the general expression used to find the number of proper factors for a specific $3^m \times 5^n$ value is,

$$m \times n + m + n - 1 \tag{10}$$

$$\Rightarrow mn + m + n - 1 \tag{11}$$

Factorising (11) into two bracketed expressions leads to,

$$(m + 1)(n + 1) - x \tag{12}$$

where x is some integer, and $+1$ is added to m and n such that $m + n$ is present when (12) is simplified. Expanding (12) and equating it to (11) gives us,

$$mn + m + n + 1 - x = mn + m + n - 1 \tag{13}$$

Since mn, m, and n are like terms that cancel each other out on LHS and RHS of (13), we are left with,

$$\cancel{mn} + \cancel{m} + \cancel{n} + 1 - x = \cancel{mn} + \cancel{m} + \cancel{n} - 1 \tag{14}$$

$$\Rightarrow 1 - x = -1 \tag{15}$$

$$\Rightarrow -x = -1 - 1 \tag{16}$$

$$\Rightarrow \cancel{-}x = \cancel{-}2 \tag{17}$$

$$\Rightarrow x = 2 \tag{18}$$

Thus, substituting (18) into (12) gives us an easier-to-solve version of (11) in the context of $3^m \times 5^n$,

$$(m + 1)(n + 1) - 2 \tag{19}$$

The total number of proper factors is 10 in this case, so equating (19) to 10 and rearranging the terms gives us,

$$(m + 1)(n + 1) - 2 = 10 \tag{20}$$

$$\Rightarrow (m + 1)(n + 1) = 12 \tag{21}$$

Given that the factors of 12 are 1, 2, 3, 4, 6, and 12, we can construct a table of possible m and n values for $3^m \times 5^n$,

$m+1$	$n+1$	m	n	$3^m \times 5^n$
1	12	0	11	$3^0 \times 5^{11}$
2	6	1	5	$3^1 \times 5^5$
3	4	2	3	$3^2 \times 5^3$
4	3	3	3	$3^3 \times 5^2$
6	2	5	1	$3^5 \times 5^1$
12	1	11	0	$3^{11} \times 5^0$

This leads to a total of 6 possible $3^m \times 5^n$ values, meaning there are 5 other integers (i.e., excluding $3^2 \times 5^3$) which satisfy the necessary properties.

4 Solution (Part III)

For the third part of the problem, our aim is to find N, the smallest positive integer with exactly 426 proper factors. As such, let p be the prime factored form of N expressed as,

$$p = 2^a \times 3^b \times 5^c \times 7^d \times \cdots \tag{22}$$

where 2, 3, 5, 7, etc. are the first few terms of the prime number sequence. Consequently, using our logic from Part II, p's number of proper factors is,

$$(a+1)(b+1)(c+1)(d+1)\cdots - 2 \tag{23}$$

where a and b (just like m and n previously) are the exponents by which each prime value is raised. Since (23) equates to 426 (i.e., p has 426 factors), rearranging the terms leads to,

$$(a+1)(b+1)(c+1)(d+1)\cdots - 2 = 426 \tag{24}$$

$$\Rightarrow (a+1)(b+1)(c+1)(d+1)\cdots = 428 \tag{25}$$

To obtain different distributions of a, b, etc. values, we factorise 428 to obtain,

$$428 = 2^2 \times 107 \tag{26}$$

with its factored forms being 428, 214×2, 107×4, and 107×2^2. This means p has different values depending on how 428's factors are distributed as exponents,

428 Factored Form	a	b	c	p
428	427	N/A	N/A	2^{428}
214×2	213	1	N/A	$2^{213} \times 3$
107×4	106	3	N/A	$2^{106} \times 3^3$
107×2^2	106	1	1	$2^{106} \times 3 \times 5$

Analysing p to find the minimum value, we first eliminate $2^4 28$ as it has the largest exponent. We are then left with $2^{213} \times 3$, $2^{106} \times 3^3$, and $2^{106} \times 3 \times 5$, which can be split up to form,

$$2^{106} \times 2^{107} \times 3 \tag{27}$$

$$2^{106} \times 27 \tag{28}$$

$$2^{106} \times 15 \tag{29}$$

Since 2^{106} is common across (27), (28), and (29), the point of comparison becomes their coefficients. $2^{107} > 27 > 15$, meaning that (29) has the smallest coefficient and, thus, overall value. Therefore, the smallest number N with 426 proper factors must be $2^{106} \times 3 \times 5$.