

1.

Show that $3^2 \times 5^3$ has exactly 10 proper factors.

We could use a table to list all the factors of $3^2 \times 5^3$.

	3^0	3^1	3^2
5^0	$3^0 \times 5^0$	$3^1 \times 5^0$	$3^2 \times 5^0$
5^1	$3^0 \times 5^1$	$3^1 \times 5^1$	$3^2 \times 5^1$
5^2	$3^0 \times 5^2$	$3^1 \times 5^2$	$3^2 \times 5^2$
5^3	$3^0 \times 5^3$	$3^1 \times 5^3$	$3^2 \times 5^3$

We can see that there is a total number of $(2 + 1) \times (3 + 1) = 12$ factors. It could be calculated by adding 1 to each of the number of powers of the two prime factors and multiply them.

And because the first factor ($3^0 \times 5^0$) and the last factor ($3^2 \times 5^3$) is always 1 and the number it self respectively, we need to subtract these 2 factors from it to get the number of proper factors. So in this case, the number of proper factors would be: $(2 + 1) \times (3 + 1) - 2 = 10$. Which proofs that $3^2 \times 5^3$ has exactly 10 proper factors.

Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

We can write a general equation using the information above for the proper factors. For all positive integer N , $N = a^m b^n$ (when a, b, m and n are all positive integers and $a \neq b, a \neq 1, b \neq 1$), the number of proper factors of N will be $(m + 1)(n + 1) - 2$.

With this formula, we can use it to present the number of proper factors of $3^m \times 5^n$, which will be $(m + 1)(n + 1) - 2 = 10$. As we know m and n are all positive integers, we can solve this equation:

$$(m + 1)(n + 1) - 2 = 10$$

$$(m + 1)(n + 1) = 12$$

$$\text{let } x = m + 1, y = n + 1,$$

$$xy = 12$$

$$\begin{cases} x_1 = 1 \\ y_1 = 12 \end{cases} \begin{cases} x_2 = 2 \\ y_2 = 6 \end{cases} \begin{cases} x_3 = 3 \\ y_3 = 4 \end{cases} \begin{cases} x_4 = 4 \\ y_4 = 3 \end{cases} \begin{cases} x_5 = 6 \\ y_5 = 2 \end{cases} \begin{cases} x_6 = 12 \\ y_6 = 1 \end{cases}$$

Therefore:

$$\begin{cases} m_1 = 0 \\ n_1 = 11 \end{cases} \begin{cases} m_2 = 1 \\ n_2 = 5 \end{cases} \begin{cases} m_3 = 2 \\ n_3 = 3 \end{cases} \begin{cases} m_4 = 3 \\ n_4 = 2 \end{cases} \begin{cases} m_5 = 5 \\ n_5 = 1 \end{cases} \begin{cases} m_6 = 11 \\ n_6 = 0 \end{cases}$$

As we know a and b are all positive integers, the final solution will be:

$$\begin{cases} m_1 = 1 \\ n_1 = 5 \end{cases} \begin{cases} m_2 = 2 \\ n_2 = 3 \end{cases} \begin{cases} m_3 = 3 \\ n_3 = 2 \end{cases} \begin{cases} m_4 = 5 \\ n_4 = 1 \end{cases}$$

$$\therefore N = 3^m \times 5^n$$

$$\therefore N_1 = 3^1 \times 5^5 = 9375, N_2 = 3^2 \times 5^3 = 1125, N_3 = 3^3 \times 5^2 = 675, N_4 = 3^5 \times 5^1 = 1215$$

So there will be three more integers of the form $3^m \times 5^n$ other than $3^2 \times 5^3$ that have exactly 10 proper factors.

2.

Let N be the smallest positive integer that has exactly 426 proper factors.

Determine N , giving your answer in terms of its prime factors.

We know from last question the formula for proper factors. In this case, we know the number of the proper factors, so the equation would be $(m + 1)(n + 1) - 2 = 426$.

Simplify this equation and we could get: $(m + 1)(n + 1) = 428$

Here we want to separate 428 into two factors. We could work out all the prime factors of 428, which is $2 \times 2 \times 107$. To make N as small as possible, we have to choose two number that multiplies to 428 and has the smallest sum. In this cause, it would be 4×107 .

$$\therefore (m + 1)(n + 1) = 428$$

$$\therefore m = 3, n = 106 \text{ or } m = 106, n = 3$$

So now we know that $N = a^3 b^{106}$ or $N = a^{106} b^3$. We want to make the smallest positive integer, so a and b could be the smallest prime factors which are $a = 2, b = 3$, and 2 goes under 106 and 3 goes under 3 to get the smallest solution.

$$\therefore N = 2^{106} \times 3^3$$