Preliminary work :

We will start by showing that if $N = {p_1}^{a_1}{p_2}^{a_2} \dots {p_n}^{a_n}$, where $p_1, p_2, \dots, p_n$ are distinct prime numbers and $a_1, a_2, \dots, a_n$ are positive integers, then N has $(1+a_1)(1+a_2)\dots(1+a_n)$ divisors.

We see that any divisor of N must be of the form ${p_1}^{b_1}{p_2}^{b_2} \dots {p_n}^{b_n}$, where $b_i \in \{0, 1, \dots, a_i\}$ for all $i = \{1, 2, \dots, n\}$.

Each value of b_i has $1 + a_i$ different possibilities. This means that for one divisor, there are $(1+a_1)(1+a_2)\dots(1+a_n)$ possibilities or N must have $(1+a_1)(1+a_2)\dots(1+a_n)$ divisors.

However, this calculation takes into account 1 and N, which are not proper factors of N. The number of proper factors of N is hence $(1+a_1)(1+a_2)\dots(1+a_n) - 2$.

Solution to #1:

We see that $3^2 \times 5^3$ has $(1+2)(1+3) - 2 = 10$ proper factors, as required.

We want to find how many other pairs (m, n) are such that $3^m \times 5^n$ has 10 proper factors. This means we want to find the number of other pairs (m, n) such that $(1+m)(1+n) - 2 = 10 \Leftrightarrow (1+m)(1+n) = 12$.

We see that $12 = 2^2 \times 3$ has $(1+2)(1+1) = 6$ divisors (including 1 and 12). This means there are 6 possibilities for $1 + m$, and each possibility gives one value of $1 + n$. Hence, there are a total of 6 pairs (m, n).

However, $(m, n) = (2, 3)$ is one of those pairs, and we are asked how many OTHER pairs there are. This means there are $6 - 1 = \boxed{5}$ other pairs. ■

Solution to #1:

Let $N = {p_1}^{a_1}{p_2}^{a_2} \dots {p_n}^{a_n}$, where $p_1, p_2, \dots, p_n$ are distinct prime numbers and $a_1, a_2, \dots, a_n$ are positive integers. Then, N has $(1+a_1)(1+a_2)\dots(1+a_n) - 2 = 426$ divisors, which means $(1+a_1)(1+a_2)\dots(1+a_n) = 428$.

We start by finding the prime factorization of 428. We obtain $428 = 2^2 \times 107$.

Since we want N to be as small as possible, we want to have the smallest exponents affecting the primes. This means each factor of the form $1 + a_i$ must be as small as possible, and the only way we can do that is by letting each prime factor of 428 be equal to one of the factors on the left-hand side of the expression. Also, since 428 has three prime factors (counting the 2 twice because of its exponent), we have $n = 3$, or $(1+a_1)(1+a_2)(1+a_3) = 428$.

We hence have:

$$\begin{cases} 1 + a_1 = 107 \Leftrightarrow a_1 = 106 \\ 1 + a_2 = 2 \Leftrightarrow a_2 = 1 \\ 1 + a_3 = 2 \Leftrightarrow a_3 = 1 \end{cases}$$

So far, we have $N = {p_1}^{106} p_2 p_3$.

Now, to make N as small as possible, we must assign the smallest primes to the largest exponents. This means we must have $(p_1, p_2, p_3) = (2, 3, 5)$ or $\boxed{N = 2^{106} \times 3 \times 5}$. ■