For $n \in \mathbb{Z}^+$, let $d(n)$ be the number of divisors of n then for

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots \alpha_k$ are positive integers.

n's divisor d is of the form:

$$d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where β_i is integer that $0 \leq \beta_i \leq \alpha_i$ for all i

so that each divisor d is correspond to a sequence $(\beta_1, \beta_2, \ldots \beta_k)$.

$\therefore$ no. divisors of n = no. possible sequences of $(\beta_1, \beta_2, \ldots \beta_k)$

$0 \leq \beta_1 \leq \alpha_1$ so there are $\alpha_1 + 1$ possible choices for β_1

$0 \leq \beta_2 \leq \alpha_2$ so there are $\alpha_2 + 1$ possible choices for β_2

s $0 \le \beta_k \le \alpha_k$ so there are $\alpha_k + 1$ possible choices for β_k

So: $d(n)$ = no. divisors of n
= no. sequences of $(\beta_1, \beta_2, \ldots, \beta_k)$
$= (\alpha_1+1)(\alpha_2+1) \ldots (\alpha_k+1)$
as disered

Then if $D(n)$ is no. divisor excluding 1 and n or no. proper factors of n then:

$$D(n) = d(n) - 2$$
$$= (\alpha_1+1)(\alpha_2+1)\ldots(\alpha_k+1) - 2.$$

1. So $D(3^2 \times 5^3) = (2+1)(3+1) - 2$
$= 12 - 2 = 10$

and for $k = 3^m \times 5^n$ that have 10 proper factors
$10 = D(k) = (m+1)(n+1) - 2$
$\Leftrightarrow (m+1)(n+1) = 12 = 2 \times 6 = 3 \times 4 = 6 \times 2 = 4 \times 3$
so $(m, n) \in \{(1,5), (2,3), (5,1), (3,2)\}$

so that there are 4 possible values of k.

2. Let $N = q_1^{k_1} q_2^{k_2} \ldots q_j^{k_j}$ then:

$426 = D(N) = (k_1+1)(k_2+1)\ldots(k_j+1) - 2$
$428 = (k_1+1)(k_2+1)\ldots(k_j+1).$
$2 \times 2 \times 107 = (k_1+1)(k_2+1)\ldots(k_j+1)$

So that $j \leq 3$ as 428 can only be expressed as the product of at most 3 positive integers greater than 1.

If $j = 3$, for N to be smallest, p_i's must be smallest so $p_1 = 2$, $p_2 = 3$, and $p_3 = 5$

Since $(k_1+1)(k_2+1)(k_3+1) = 2 \times 2 \times 107$,

$$(k_1, k_2, k_3) \in \{(106, 1, 1), (1, 106, 1), (1, 1, 106)\}$$

$$N \in \{2^{106} \times 3 \times 5;\ 2 \times 3^{106} \times 5;\ 2 \times 3 \times 5^{106}\}$$

Clearly: $2 \times 3 \times 5^{106} > 2 \times 3^{106} \times 5 > 2^{106} \times 3 \times 5$

so for this case, $N = 2^{106} \times 3 \times 5$

If $j = 2$, for N to be smallest, $p_1 = 2$ and $p_2 = 3$ and:

$$\begin{aligned}(k_1+1)(k_2+1) &= 107 \times 2 \times 2 \\ &= 214 \times 2 \\ &= 107 \times 4\end{aligned}$$

so that: $(k_1, k_2) \in \{(213, 1); (1, 213); (106, 3); (3; 106).\}$

and $N \in \{2^{213} \times 3;\ 2 \times 3^{213};\ 2^{106} \times 3^3;\ 2^3 \times 3^{106}\}$

Now again, clearly: $2^{213} \times 3 < 2 \times 3^{213}$

and: $2^{106} \times 3^3 < 2^3 \times 3^{106}$

→ To find smallest N in this case, we compare $2^{213} \times 3$ and $2^{106} \times 3^3$. Consider:

$$\frac{2^{213} \times 3}{2^{106} \times 3^3} = \frac{2^{213-106}}{3^{3-2}} = \frac{2^{107}}{3} = \frac{4}{3} \times 2^{105} > 3$$

So $2^{106} \times 3^3 < 2^{213} \times 3$ and so for this case,

$N = 2^{\text{[illegible]}} \times 3$

If $j = 1$, again, $p_1 = 2$ and: $k_1 + 1 = 428$

so that $k_1 = 427$ and $N = 2^{427}$

Overall, smallest values of N are now:

$$2^{106} \times 3 \times 5, \quad 2^{106} \times 3^3 \text{ and } 2^{427}.$$

Note that:

$$\frac{2^{427}}{2^{106} \times 3^3} = \frac{2^{427-106}}{3^3} = \frac{2^{321}}{27}$$

$$= \frac{2^5 \times 2^{316}}{27}$$

$$= \frac{32}{27} \times 2^{316} > 1$$

and:

$$\frac{2^{106} \times 3^3}{2^{106} \times 3 \times 5} = \frac{9}{5} > 1$$

$$\therefore \quad 2^{427} > 2^{106} \times 3^3 > 2^{106} \times 3 \times 5$$

$\therefore$ The smallest possible N is $\boxed{2^{106} \times 3 \times 5}$