Divisible Factorisations

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Age 16 to 18
Challenge Level ★

Let n be a positive integer.

1. Factorise $n^5 - n^3$, and show that it is divisible by 24.
2. Prove that $2^{2n} - 1$ is divisible by 3.
3. If $n - 1$ is divisible by 3, show that $n^3 - 1$ is divisible by 9.

1. $n^5 - n^3 = n^3(n^2 - 1)$

$= n^3(n+1)(n-1)$

$24 = 3 \times 8$

$n^5 - n^3$ is divisible by 3 because $(n-1), n, (n+1)$ are three consecutive integers, and between them there must be at least one number which is a multiple of 3.

$n^5 - n^3$ is divisible by 8 because:

if n is odd, $n = 2x + 1$

$n^2 - 1 = (2x+1)^2 - 1$

$= 4x^2 + 4x + 1 - 1$

$= 4x^2 + 4x$

$= 4x(x+1)$ ⟶ This expression will always be divisible by 8 because:

if x is odd:

$4x(\text{odd} + 1) = 4x \cdot \text{even}$

$4 \times \text{even} = \text{multiple of } 8$

if x is even:

$4 \times \text{even} \times (x+1)$

$4 \times \text{even} = \text{multiple of } 8.$

if n is even, $n = 2x$

$n^3 = (2x)^3$

$= 8x^3$ ⟶ This is a multiple of 8

$\therefore n^5 - n^3$ is always divisible by 24, because it is divisible by both 3 & 8

2. $2^{2n} - 1$ is divisible by 3.

Factorise $(2^n + 1)(2^n - 1)$ ⟶ $(2^n - 1), 2^n, (2^n + 1)$ ⟶ Those are three consecutive integers

2^n is the number in between the factors

2^n is always even, and only has factor of 2. $2^n + 1$ is an odd number, $2^n - 1$ is also an odd number. With similar reason, because one of the integer of three consecutive integers has to be the multiple of 3, and 2^n is not a multiple of 3, therefore one of $2^n + 1$ and $2^n - 1$ has to be the multiple of 3. Therefore, divisible by 3.

3. $n-1$ is divisible by 3, n^3-1 divisible by 9

$$(n-1)^3 = (n^2-2n+1)(n-1)$$

$$= n^3-2n^2+n-n^2+2n-1$$

$$= n^3-3n^2+3n-1$$

This expression is divisible by 27, because we cubed $(n-1)$, which has a factor of 3, $3^3 = 27$

$$n^3-3n^2+3n-1$$

$$= n^3-1-(3n^2-3n)$$

$$= n^3-1-(3n(n-1))$$

Because $n-1$ is divisible by 3, $3n(n-1)$ is divisible by 9.

(n^3-1) − (multiple of 9) = multiple of 27

n^3-1 = multiple of 27 + multiple of 9

n^3-1 = 3 × (multiple of 9) + multiple of 9

(still a multiple of 9)

n^3-1 must be a multiple of 9

$\therefore$ n^3-1 is divisible by 9