1. I am sure there's perhaps a faster way of doing this but this is the solution I found. Firstly, we can attempt a little prime factorisation to notice that $24 \mid a \iff 2^3, 3 \mid a$. So, we can develop a two pronged attack onto this proof - one to prove that our expression is divisible by 8 and one to show it is divisible by 9.

   Factorising $n^5 - n^3$ we get $n^3(n+1)(n-1)$. Slipping in an n we get $n^2((n-1)(n)(n+1))$. That inside bracket is the product of three consecutive numbers which is always divisible by three. This can be done quite easily if you think about it, a multiple of 3 occurs every 3 numbers thus over the span of 3 consecutive numbers at least one of them is a multiple of 3 thus the product of those consecutive numbers must be divisible by 3.

   To prove everything was divisble by 8, I took quite the boring method. Case one is that n was even $\Rightarrow n = 2q,\ q \in \mathbb{N} \Rightarrow (2q)^3((4q^2) - 1) = 8q^3(...)$ thus it holds for even values. If n is odd $\Rightarrow n = 2q+1 \Rightarrow (2q+1)^3(2q+2)(2q) = 8(4q^5+10q^4+9q^3+28q^2+4q)+28q^2+4q = 8(r)+4(7q^2+1)$, $r \in \mathbb{N}$.We use a very similar idea to prove that $4(7q^2 + 1)$ is always divisible by 8 which is simple as all we need to really prove is $(7q^2 + 1)$ is always even and then factor out a 2 to get an 8. If q is even we get odd + 1 = even and if q is odd we get the same result.

   Thus $24 \mid n^5 - n^3,\ \forall n \in \mathbb{N}$

2. This proof can be done quite nicely via induction (a rather lovely tool). First, we assume a base case - for this question, our base case is $n = 1$ since our statement holds for all positive integers. We can then attempt to prove our base case which is rather simple for this question. $2^{2(1)} - 1 = 3$.

   Now, we take a rather bold move where we assume that $3 \mid 2^{2k} - 1$ for some $k \in \mathbb{N} \Rightarrow 2^{2k} - 1 = 3q$ for some $q \in \mathbb{Z}$. The notation may look intimidating at first if you haven't seen it before but all it says that the $2^{2k} - 1$ is divisible by 3 which means it can be written as $3q$ where q is an integer.

   This may seem pointless at first but it will be helpful when analyse the next case for $n = k+1$, $2^{2(k+1)} - 1$. We can use the law of indices to help re write our expression as $2^2 \cdot 2^k - 1$. Now, we simply factor out a 4 but naturally you may be scared to do so because we can't easily pull out a 4 from that -1 can we ? Well say we did anyways, and we had $4(2^k - 1)$ and we expanded it all out, we'd get $4 \cdot 2^k - 4$ which is almost what we had before ! Ah-ha, you might notice all we have to do is add three and we get back what we had. So, for $n = k+1$ we get, $4(2^k - 1) + 3$

   This is where the magic happens. Remember that almost random step we did in the 2nd paragrah - well look inside that bracket, it's our case for $n = k$ so we can rewrite it again as $4(3q) + 3$ which is simply $3(4q + 1)$.

   This is when we have to use some logic, if we proved the statement to be true for $n = 1$ and we assumed it to be true for $n = k$ and using that result we proved that it's true for $n = k + 1$ then it must be true $\forall n \in \mathbb{N}$

3. Firstly, we can notice that if $3 \mid n - 1 \Rightarrow n - 1 = 3q,\ q \in \mathbb{Z}$. Thus, $n = 3q + 1$. If we attempt to directly plug this into our $n^3 - 1$ We get $(3q + 1)^3 - 1$ which if we expand it all out gives us $27n^3 + 27n^2 + 9n + 1 - 1$ then $9(3n^3 + 3n^2 + 1)$ which is clearly divisible by 9.