Solution to #1:

Notice $n^5 - n^3 = n^3(n^2 - 1) = n^3(n-1)(n+1)$. Notice also that $24 = 2^3 \times 3$. This means our expression must be divisible by 2^3 and by 3.

Let $f(n) = \frac{n^3(n-1)(n+1)}{24}$.

Case 1: n is even ($n = 2k$ for some $k \in \mathbb{N}$)

We see that $f(2k) = \frac{(2k)^3(2k-1)(2k+1)}{24} = \frac{k^3(2k-1)(2k+1)}{3}$.

If k is a multiple of 3, then our expression is obviously an integer.
If $k = 3m + 1$ for some $m \in \mathbb{N}$, then $2k + 1 = 2(3m + 1) + 1 = 6m + 3$, which is a multiple of 3. Since $2k + 1$ is a multiple of 3, our expression is an integer.
If $k = 3m - 1$ for some $m \in \mathbb{N}$, then $2k - 1 = 2(3m - 1) - 1 = 6m - 3$, which is a multiple of 3. Since $2k - 1$ is a multiple of 3, our expression is an integer.

Case 2: n is odd ($n = 2k + 1$ for some $k \in \mathbb{N}$)

We see that $f(2k+1) = \frac{(2k+1)^3\big((2k+1)-1\big)\big((2k+1)+1\big)}{24} = \frac{k(k+1)(2k+1)^3}{6}$.

Since we want the numerator to be divisible by 6, it must be divisible by 2 and 3 since $6 = 2 \times 3$.

Notice that k and $k + 1$ are two consecutive integers, which means one of them is bound to be even. This means the numerator is divisible by 2, and it remains to show that the numerator is divisible by 3.

If k is a multiple of 3, then our expression is obviously an integer.
If $k = 3m + 1$ for some $m \in \mathbb{N}$, then $2k + 1 = 2(3m + 1) + 1 = 6m + 3$, which is a multiple of 3. Since $2k + 1$ is a multiple of 3, our expression is an integer.
If $k = 3m - 1$ for some $m \in \mathbb{N}$, then $k + 1 = (3m + 1) + 1 = 3m$, which is a multiple of three. Since $k + 1$ is a multiple of 3, our expression is an integer.

In all cases, $f(n)$ is an integer, which proves our statement. ■

Solution to #2:

We see that $2^{2n} - 1 = (2^n)^2 - 1 = (2^n - 1)(2^n + 1)$. If $2^{2n} - 1$ is divisible by 3, then one of $2^n - 1$ and $2^n + 1$ must be divisible by 3.

This means we want to show that for every $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $2^n = 3k \pm 1$. We will prove this by induction.

Let $S(n)$ be the statement $2^n = 3k \pm 1$.

Base case: We can see that $2^1 = 3(1) - 1$, which verifies $S(1)$.

Induction step: Suppose there exists some $m \in \mathbb{N}$ such that $S(m)$ is true, i.e. $2^m = 3k \pm 1$ for some k. We see that:

$$2^m = 3k \pm 1 \Longrightarrow 2^{m+1} = 6k \pm 2 = 6k \pm (3 - 1) = 6k \pm 3 \mp 1 = 3(2k + 1) \mp 1$$

We see that $2^{m+1} = 3(2k + 1) \mp 1$ verifies $S(m + 1)$, which completes the induction step.

Hence, $S(n)$ holds for all natural numbers n, which means that $2^{2n} - 1$ is divisible by 3 for all natural numbers n. ■

Solution to #3:

If $n - 1$ is divisible by 3, then $n = 3k + 1$ for some $k \in \mathbb{N}$. This means

$$\begin{aligned} n^3 - 1 = (3k + 1)^3 - 1 &= (27k^3 + 27k^2 + 9k + 1) - 1 = 27k^3 + 27k^2 + 9k \\ &= 9(3k^3 + 3k^2 + k) \end{aligned}$$

This means $(3k + 1)^3 - 1$ is always divisible by 9, which proves our statement. ■