1. I will lay out some basic proofs to aid later proofs!

$2K$ is an even number and any number in the form $2(K)$ is also even

$2K+1$ is an odd number and any number in the form $2(K)+1$ is also odd.

→ $K$ is some integer.

1) Even × Even = Even

$(2n) \times (2K) = 4nK$

$= 2(2nK)$ → in the form $2(K)$

2) Odd × Odd = Odd

$(2n+1)(2K+1) = 4Kn + 2n + 2K + 1$

$= 2(2Kn + n + K) + 1$ → in form $2(K)+1$

3) Odd × Even = Even

$(2n+1)(2K) = 4Kn + 2K$

$= 2(2Kn + K)$ → in form $2(K)$

4) Odd + Even = Odd

$2n+1 + 2K = 2(n+K)+1$ → in form $2(K)+1$

5) Odd + odd = Even

$2n+1 + 2K+1 = 2n + 2K + 2$

$= 2(n+K+1)$ → in form $2(K)$

6) Even + Even = Even

$2n + 2K = 2(K+n)$ → in form $2(K)$.

NOTE: $n$ is also some random integer.

$n^5 - n^3$ can be factorised as:

$n^3(n^2-1)$

$\rightarrow n^3(n+1)(n-1)$

WE NOTE THAT TO BE A MULTIPLE OF 24, the number must have prime factors of $2^3 \times 3$.

WE ALSO NOTE THAT SINCE $n, n+1, n-1$ are consecutive, at least one off $n, n+1$ and $n-1$ is a multiple of three.

There are 2 cases; when $n$ is even and when $n$ is odd.

Case1 $\rightarrow$ n is even.

If $n$ is even it can be written in the form $2K$

Therefore, let $n = 2K$

Then the expression is:

$$(2K)^3(2K+1)(2K-1) = 8K^3(2K+1)(2K-1)$$

Since $(2K)^3$ is $8K^3$, we have a factor of $2^3$, 8

Therefore, since one of $2K$, $2K+1$ or $2K-1$ is a multiple of 3, the expression must be divisible by at least $2^3 \times 3$. Therefore, when $n$ is even, $n^5 - n^3$ is divisible by 24 when $n$ is even.

Case 2 $\rightarrow$ n is odd.

If this case we let $n = 2K+1$

Let

$n = 2K+1$

Then the expression becomes:

$$(2K+1)^3(2K+2)(2K).$$

This can be expressed as:

$(2K+1)^3(4K^2+4K)$

$= (2K+1)^3 4(K^2+K)$

We already know that one ~~of~~ $2K+1$, $2K+2$ or $2K$ is a multiple of 3 since they are consecutive. Therefore, we have a factor of 3.

Also in step (*) we have established that there is a factor of $2^2$ (4).

Therefore, we currently have $2^2 \times 3$.

Finally, using proofs ~~1, 2, 4, 5~~ 1, 2, 5, 6 we establish:

Case K is even

If K is even, then $K^2$ is even (proof 1).
Then $K^2+K$ is even (proof 6)

Case K is odd:

If K is odd then $K^2$ is odd (proof 2)
Then $K^2+K$ is even (proof 5).

In both cases, $K^2+K$ is even so can be written in the form $2n$ where n is some number. Therefore, ~~so~~ the factors can be multiplied by 2:
$2^2 \times 3 \quad \times 2 \quad = 2^3 \times 3$.

Therefore, $n^5 - n^3$ is divisible by $2^3 \times 3$ which is 24. when n is odd.
THEREFORE IN BOTH CASES (O and E) $n^5-n^3$ is divisible by 24.

4

2) $2^{2n}$ can be written as $4^n$ by laws of indices:

$$2^{2n} = (2^2)^n = 4^n$$

$4^n$ can be written as $(3+1)^n$.

The entire expression, as a result can be written as $(3+1)^n - 1$

By binomially expanding $(3+1)^n$ we have

NOTE : $(a+b)^n \to (3+1)^n$ $a=3$, $b=1$

$${}^nC_0\, 3^n 1^0 + {}^nC_1\, 3^{n-1} b^1 \ldots {}^nC_n\, 3^0 1^n$$

In this case, $n \neq 0$ therefore, all terms except the last term are divisible by 3 since they have a factor of $3^n$, $3^{n-1}$, $3^{n-2}$ ...

The last term, ${}^nC_n\, 3^0 1^n$ however will always be equal to 1.

$\to$ ${}^nC_n$ is always equal to 1, by Pascals triangle:

1 $\to$

1 1 $\to$

1 2 1 $\to$ ALWAYS EQUAL TO 1.

1 3 3 1 $\to$

$\to$ $3^0$ is equal to 1

$\to$ $1^n$ is always equal to 1.

Therefore, every term except the last term is a multiple of three.

Adding each term, except the last will give a MULTIPLE of 3 since multiple of threes added together gives a multiple of 3:

$3n + 3K = 3(n+K)$.

Therefore, the overall expansion can be expressed as $3(Z)+1$ where $Z$ is some integer.

As a result, subtracting the 1 gives:

$3(Z)+1-1$

$\Rightarrow 3(Z)$

This leaves behind a multiple of 3.

Therefore, $2^{2n}-1$ is ALWAYS A MULTIPLE OF 3 when $n$ is a positive integer (in the question).

3) $\frac{n-1}{3} = K$ where $K$ is some integer.

$n-1 = 3K$

$n = 3K+1$

$\therefore n^3 - 1 = (3K+1)^3 - 1$

$= (27K^3 + 9K^2 + 18K^2 + 6K + 3K + 1) - 1$

$= (27K^3 + 27K^2 + 9K + 1) - 1$

$= 9(3K^3 + 3K^2 + K)$.

Therefore, as the +1 and −1 cancel out, we are left with a multiple of 9.

Therefore, $n^3-1$ is divisible by 9 when $n-1$ is divisible by 3.