Square Difference (NRICH Solution)

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1 Abstract

This document presents a solution to the "Square Difference" problem published by the University of Cambridge's NRICH website. Methods used throughout this proof include the application of common number theory principles (e.g., proof by example) alongside basic numeric and algebraic manipulation. I hope my explanations will help budding mathematicians better understand the intricacies behind solving this problem!

2 Solution (Part I)

For the first part of the problem, our aim is to express the numbers,

$$3, 5, 8, 12, 16 \tag{1}$$

as the difference of two non-zero squares. Given that the first 5 squares are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, and $5^2 = 25$, we can express each value in (1) using these squares,

$$3 = 4 - 1 \tag{2}$$

$$5 = 9 - 4 \tag{3}$$

$$8 = 9 - 1 \tag{4}$$

$$12 = 16 - 4 \tag{5}$$

$$16 = 25 - 9 \tag{6}$$

This gives us the following equations in terms of squares for each value in the list,

$$3 = 2^2 - 1^2 \tag{7}$$

$$5 = 3^2 - 2^2 \tag{8}$$

$$8 = 3^2 - 1^2 \tag{9}$$

$$12 = 4^2 - 2^2 \tag{10}$$

$$16 = 5^2 - 3^2 \tag{11}$$

3 Solution (Part II)

For the second part of the problem, our aim is to show that any odd number can be expressed as the difference of two squares. Consequently, let the two squares be consecutive, with their subtraction written as either,

$$(m+1)^2 - m^2 \tag{12}$$

or,

$$n^2 - (n-1)^2 \tag{13}$$

where m and n are distinct odd or even integers (i.e., each value's parity doesn't affect the end outcome.) Firstly, expanding (12) gives us,

$$m^2 + 2m + 1 - m^2 \tag{14}$$

$$\Rightarrow \cancel{m^2} + 2m + 1 - \cancel{m^2} \tag{15}$$

$$\Rightarrow 2m + 1 \tag{16}$$

Likewise, expanding (13) leads to,

$$n^2 - (n^2 - 2n + 1) \tag{17}$$

$$\Rightarrow n^2 - n^2 + 2n - 1 \tag{18}$$

$$\Rightarrow \cancel{n^2} - \cancel{n^2} + 2n - 1 \tag{19}$$

$$\Rightarrow 2n - 1 \tag{20}$$

Therefore, regardless of how the difference between each square is structured and whether m or n is odd or even, the final value is always odd. This can be proven using examples of odd and even input values for (16) and (20),

m	n	$2m+1$	$2n-1$
1	1	3	1
2	2	5	3
3	3	7	5
4	4	9	7
5	5	11	9

This proves that all odd numbers can be written as the difference of two squares. (Q.E.D.)

4 Solution (Part III)

For the third part of the problem, our aim is to show that any number of the form $4k$ (where k is a non-negative integer) can be expressed as the difference of two squares. Splitting $4k$ into two values produces the possibilities of,

$$k + 3k = 4k \tag{21}$$

$$2k + 2k = 4k \tag{22}$$

$$3k + k = 4k \tag{23}$$

By letting k be the variable used in (16) and (20) from Part II, and using the combination logic of (22), we get the expression,

$$2k + 1 + 2k - 1 \tag{24}$$

$$\Rightarrow 2k + \cancel{1} + 2k - \cancel{1} \tag{25}$$

$$\Rightarrow 4k \tag{26}$$

This means that (24) can be expressed by adding together two distinct square difference expressions, namely (12) and (13), in terms of k,

$$[(k+1)^2 - k^2] + [k^2 - (k-1)^2] \tag{27}$$

$$\Rightarrow (k+1)^2 - \cancel{k^2} + \cancel{k^2} - (k-1)^2 \tag{28}$$

$$\Rightarrow (k+1)^2 - (k-1)^2 \tag{29}$$

$$\Rightarrow k^2 + 2k + 1 - (k^2 - 2k + 1) \tag{30}$$

$$\Rightarrow \cancel{k^2} + 2k + \cancel{1} - \cancel{k^2} + 2k - \cancel{1} \tag{31}$$

$$\Rightarrow 2k + 2k = 4k \tag{32}$$

As such, we see a difference of two squares being created in (29) which simplifies to 4k when the numbers being squared are two apart. This is true since,

$$k + 1 - (k - 1) \tag{33}$$

$$\Rightarrow \cancel{k} + 1 - \cancel{k} + 1 \tag{34}$$

$$\Rightarrow 1 + 1 = 2 \tag{35}$$

This proves that all numbers of the form $4k$ can be written as the difference of two squares, as long as a separation of two is present between each value being squared. (Q.E.D.)

5 Solution (Part IV)

For the fourth part of the problem, our aim is to show that no number of the form $4k+2$ (where k is a non-negative integer) can be expressed as the difference of two squares. For positive integers a and b in,

$$a^2 - b^2 \tag{36}$$

we see that a parity approach for the input values yields 4 possible outcomes,

$$a = \text{odd}, b = \text{even} \tag{37}$$

$$\Rightarrow a = 2m + 1, b = 2n \tag{38}$$

$$a = \text{even}, b = \text{odd} \tag{39}$$

$$\Rightarrow a = 2m, b = 2n + 1 \tag{40}$$

$$a = \text{odd}, b = \text{odd} \tag{41}$$

$$\Rightarrow a = 2m + 1, b = 2n + 1 \tag{42}$$

$$a = \text{even}, b = \text{even} \tag{43}$$

$$\Rightarrow a = 2m, b = 2n \tag{44}$$

where m and n are positive integers (i.e., odd or even). Thus, substituting the algebraic values of (38), (40), (42), and (44) gives us,

a	b	$a^2 - b^2$	$a^2 - b^2$ Simplified	$a^2 - b^2$ Parity
$2m + 1$	$2n$	$(2m + 1)^2 - (2n)^2$	$4(m^2 - n^2 + m) + 1$	Odd
$2m$	$2n + 1$	$(2m)^2 - (2n + 1)^2$	$4(m^2 - n^2 - n) - 1$	Odd
$2m + 1$	$2n + 1$	$(2m + 1)^2 - (2n + 1)^2$	$4(m^2 - n^2 + m - n)$	Even
$2m$	$2n$	$(2m)^2 - (2n)^2$	$4(m^2 - n^2)$	Even

This proves that all the even possibilities for the difference of two squares (i.e., odd^2 or even^2) are of the form $4k$, not $4k+2$, as they are multiples of 4. (Q.E.D.)

6 Solution (Part V)

For the fifth part of the problem, our aim is to show that any number of the form pq (where p and q are prime numbers > 2) can be expressed as the difference of two squares in two different ways. Using integers m and n, we get the equation,

$$m^2 - n^2 = pq \tag{45}$$

meaning that pq can be broken down to form,

$$(m + n)(m - n) = pq \tag{46}$$

Splitting (46) by factorisation yields two possible cases, where we let $p \geq q$ and both values be odd (i.e., since all prime numbers > 2 are odd). Firstly,

$$(m + n) = pq \tag{47}$$

$$(m - n) = 1 \tag{48}$$

and secondly,

$$(m + n) = p \tag{49}$$

$$(m - n) = q \tag{50}$$

This gives us two methods for writing square differences using pq. But to prove the equations' validity, we express m by adding (47) and (48) for case 1, (49) and (50) for case 2 whilst n is expressed by subtracting the said equations,

$$(m+n)+(m-n)=pq+1 \tag{51}$$

$$\Rightarrow m+\cancel{n}+m-\cancel{n}=pq+1 \tag{52}$$

$$\Rightarrow 2m=pq+1 \tag{53}$$

$$\Rightarrow m=\frac{pq+1}{2} \tag{54}$$

$$(m+n)-(m-n)=pq-1 \tag{55}$$

$$\Rightarrow \cancel{m}+n-\cancel{m}+n=pq-1 \tag{56}$$

$$\Rightarrow 2n=pq-1 \tag{57}$$

$$\Rightarrow n=\frac{pq-1}{2} \tag{58}$$

$$(m+n)+(m-n)=p+q \tag{59}$$

$$\Rightarrow m+\cancel{n}+m-\cancel{n}=p+q \tag{60}$$

$$\Rightarrow 2m=p+q \tag{61}$$

$$\Rightarrow m=\frac{p+q}{2} \tag{62}$$

$$(m+n)-(m-n)=p-q \tag{63}$$

$$\Rightarrow \cancel{m}+n-\cancel{m}+n=p-q \tag{64}$$

$$\Rightarrow 2n=p-q \tag{65}$$

$$\Rightarrow n=\frac{p-q}{2} \tag{66}$$

Substituting (54) and (58) into (45) gives us,

$$(\frac{pq+1}{2})^2-(\frac{pq-1}{2})^2 \tag{67}$$

$$\Rightarrow (\frac{p^2q^2+2pq+1}{4})-(\frac{p^2q^2-2pq+1}{4}) \tag{68}$$

$$\Rightarrow \frac{\cancel{p^2q^2}+2pq+\cancel{1}-\cancel{p^2q^2}+2pq-\cancel{1}}{4} \tag{69}$$

$$\Rightarrow \frac{4pq}{4}=pq \tag{70}$$

Likewise, substituting (62) and (66) into (45) leads to,

$$(\frac{p+q}{2})^2-(\frac{p-q}{2})^2 \tag{71}$$

$$\Rightarrow (\frac{p^2 + 2pq + q^2}{4}) - (\frac{p^2 - 2pq + q^2}{4}) \tag{72}$$

$$\Rightarrow \frac{\cancel{p^2} + 2pq + \cancel{q^2} - \cancel{p^2} + 2pq - \cancel{q^2}}{4} \tag{73}$$

$$\Rightarrow \frac{4pq}{4} = pq \tag{74}$$

Since both sets of input values for m and n lead to the same result, a square difference must be represented using precisely two distributions of p and q. This can be further proven by analysing when the two factorisations are identical,

$$\frac{pq + 1}{\cancel{2}} = \frac{p + q}{\cancel{2}} \tag{75}$$

$$\Rightarrow pq + 1 = p + q \tag{76}$$

$$\Rightarrow pq - p = q - 1 \tag{77}$$

$$\Rightarrow p(q - 1) = 1(q - 1) \tag{78}$$

$$\Rightarrow (p - 1)(q - 1) = 0 \tag{79}$$

meaning that either $p = 1$ or $q = 1$ (i.e., two possibilities). Additionally, using proof by example with p and q input values in (54), (58), (62), and (66) bolsters our conclusion. For $p = 5$, $q = 3$, and $pq = 15$,

$$\Rightarrow m = \frac{(5 \times 3) + 1}{2} \tag{80}$$

$$\Rightarrow m = 8 \tag{81}$$

$$\Rightarrow n = \frac{(5 \times 3) - 1}{2} \tag{82}$$

$$\Rightarrow n = 7 \tag{83}$$

and secondly,

$$\Rightarrow m = \frac{5 + 3}{2} \tag{84}$$

$$\Rightarrow m = 4 \tag{85}$$

$$\Rightarrow n = \frac{5 - 3}{2} \tag{86}$$

$$\Rightarrow n = 1 \tag{87}$$

Substituting (81) and (83) in (45) gives us,

$$8^2 - 7^2 = 64 - 49 \tag{88}$$

$$\Rightarrow 15 = pq \tag{89}$$

Likewise, substituting (85) and (87) in (45) leads to,

$$4^2 = 1^2 = 16 - 1 \tag{90}$$

$$\Rightarrow 15 = pq \tag{91}$$

These results allow us to conclude that exactly two distinct ways exist for writing the difference of two squares from pq. (Q.E.D.) However, note that for $p > 2$ and $q = 2$ (where both are prime, meaning $p =$ odd),

$$m^2 - n^2 = 2p \tag{92}$$

Using examples of several p input values, $2p$ can be shown to always be of the form $4k + 2$ instead of $4k$ (which, as shown in Part IV, is necessary for a square difference expression), meaning no m and n values satisfy (92),

$$p = 3 \tag{93}$$

$$\Rightarrow 2p = 6 = 4(1) + 2 \tag{94}$$

$$p = 5 \tag{95}$$

$$\Rightarrow 2p = 10 = 4(2) + 2 \tag{96}$$

$$p = 7 \tag{97}$$

$$\Rightarrow 2p = 14 = 4(3) + 2 \tag{98}$$

$$p = 11 \tag{99}$$

$$\Rightarrow 2p = 22 = 4(5) + 2 \tag{100}$$

This proves that either p or q being 2 prevents a difference of two squares from forming, whereby both must instead be odd numbers.

7 Solution (Part VI)

For the sixth part of the problem, our aim is to find how many different ways 675 can be expressed as the difference of two squares. Prime factorising 675 to calculate its number of factors (where a and b are positive integers) gives us,

$$675 = 27 \times 25 \tag{101}$$

$$\Rightarrow 675 = 3^a \times 5^b \tag{102}$$

$$\Rightarrow a = 3, b = 2 \tag{103}$$

Since $(a+1)(b+1)-2$ represents a value's number of proper factors (as shown in "Proper Factors" NRICH solution from May, 2023), the total number of factors must be,

$$(a + 1)(b + 1) \tag{104}$$

since we are not excluding 1 and the value itself. Therefore, 675's total number of factors is,

$$(3 + 1)(2 + 1) = 12 \tag{105}$$

whereby the factors themselves (in ascending order) are,

$$3^0, 3^1, 5^1, 3^2, 3 \times 5, 5^2, 3^3, 3^2 \times 5, 3 \times 5^2, 3^3 \times 5, 3^2 \times 5^2, 3^3 \times 5^2 \tag{106}$$

$$\Rightarrow 1, 3, 5, 9, 15, 25, 27, 45, 75, 135, 225, 675 \tag{107}$$

This leads to $\frac{12}{2} = 6$ factor-pairs, which can be found by multiplying the smallest and largest numbers in order from left and right,

$$1 \times 675, 3 \times 225, 5 \times 135, 9 \times 75, 15 \times 45, 25 \times 27 \tag{108}$$

Each pair includes respective p and q values, allowing for the m and n in (45) to be calculated,

$$m = \frac{675 + 1}{2} = 338, n = \frac{675 - 1}{2} = 337 \tag{109}$$

$$\Rightarrow (338 + 337)(338 - 337) = 675 \tag{110}$$

$$m = \frac{225 + 3}{2} = 114, n = \frac{225 - 3}{2} = 111 \tag{111}$$

$$\Rightarrow (114 + 111)(114 - 111) = 675 \tag{112}$$

$$m = \frac{135 + 5}{2} = 70, n = \frac{135 - 5}{2} = 65 \tag{113}$$

$$\Rightarrow (70 + 65)(70 - 65) = 675 \tag{114}$$

$$m = \frac{75 + 9}{2} = 42, n = \frac{75 - 9}{2} = 33 \tag{115}$$

$$\Rightarrow (42 + 33)(42 - 33) = 675 \tag{116}$$

$$m = \frac{45 + 15}{2} = 30, n = \frac{45 - 15}{2} = 15 \tag{117}$$

$$\Rightarrow (30 + 15)(30 - 15) = 675 \tag{118}$$

$$m = \frac{27 + 25}{2} = 26, n = \frac{27 - 25}{2} = 1 \tag{119}$$

$$\Rightarrow (26 + 1)(26 - 1) = 675 \tag{120}$$

This leads to a total of 6 distinct ways in which 675 can be written as the difference of two squares.