1. $3 = 2^2 - 1^2$

$5 = 3^2 - 4^2$

$8 = 3^2 - 1^2$

$12 = 4^2 - 2^2$

$16 = 5^2 - 3^2$

2. Any odd numbers are of the form $2n+1$ where $n \in \mathbb{N}$. Since:

$$2n+1 = n^2+2n+1-n^2$$
$$= (n+1)^2 - n^2$$

it is easy to see that all odd numbers can be expressed as the difference of 2 squares

3. Note that for any k:

$$4k = (k^2+2k+1) - (k^2-2k+1)$$
$$= (k+1)^2 - (k-1)^2$$

Hence, any number of the form $4k$, where $k \in \mathbb{N}$ can be written as the diffrence of 2 squares

4. Let k be an abitrary non-negative integer. Assume that $4k+2$ can be written as the difference of 2 squares, that is there exist $a, b \in \mathbb{N}$ such that:

$$4k+2 = a^2 - b^2$$

So $\quad 4k+2 = (a+b)(a-b).$ (1)

Note that $(a+b + a-b) = 2a$, which is an even number so both $(a+b)$ and $(a-b)$ must be either odd or even.

From identity (1), $(a+b)(a-b) = 4k+2$ is an even number so either $(a+b)$ or $(a-b)$ must be even, meaning that both of them are even. So there exist $a', b' \in \mathbb{N}$ such that:

$$a+b = 2a'$$
$$a-b = 2b'$$

which lead to:

$$4k+2 = (a+b)(a-b) = 4a'b'$$

$$\rightarrow \quad 4 \mid 4k+2 \quad (\text{impossible})$$

So that the assumption is wrong. Therefore for any $k \in \mathbb{N}$, $4k+2$ cannot be expressed as the diffrence of 2 squares.

For questions 5 and 6, let us prove the following claim:
Let $d(n)$ be the number of divisors of a positive integer n. Then for any odd non-square number, the number of ways it can be expressed as difference of 2 squares, $N(n)$ is:

$$N(n) = \frac{d(n)}{2}.$$

Proof:
Let m be an abitrary odd non-square number. If m can be written as $m = a^2 - b^2$, then there exist an ordered pair of odd numbers (x, y) such that:

$$m = xy \quad \text{and } x > y$$

$(x = a+b$ and $y = a-b)$

Likewise, if m can be expressed as $m = xy$ with x, y are odd numbers and $x > y$, there exist an ordered pair of whole number (a, b) such that:

$$m = a^2 - b^2$$

$$\left(a = \frac{x+y}{2},\ b = \frac{x-y}{2}\right)$$

So: $N(m)$ = number of ways m can be expressed as $m = xy$ with x, y: odd and $x > y$

$$= \left|\{(x,y) \mid m = xy,\ x,y: \text{odd},\ x > y\}\right|$$

$$= \left|\{(x, \tfrac{m}{x}) \mid x: \text{odd},\ x > \tfrac{m}{x},\ x \mid m\}\right|$$

$$= \left|\{(x; \tfrac{m}{x}) \mid x: \text{odd},\ x > \sqrt{m},\ x \mid m\}\right|$$

$$= \left|\{x \mid x > \sqrt{m},\ x: \text{odd}\},\ x \mid m\}\right|$$

= no. divisors of m that is greater than $\sqrt{m}$. $(*)$

Note also that if d is a divisor of m with $d > \sqrt{m}$, $\frac{m}{d}$ is also divisor of m and $\frac{m}{d} < \sqrt{m}$

Likewise, if d' is a divisor of m with $d' < \sqrt{m}$, $\frac{m}{d'}$ is also divisor of m and $\frac{m}{d'} > \sqrt{m}$

So that: no. divisors of m that is greater than $\sqrt{m}$

= no. divisors of m that is smaller than $\sqrt{m}$

Now since m is not a square, m has no divisor equals to $\sqrt{m}$ so:

no. divisors of m greater than $\sqrt{m}$

+ no. divisors of m smaller than $\sqrt{m}$

$= d(m)$.

or: no. divisors of m greater than $\sqrt{m}$ $= \dfrac{d(m)}{2}$ (**)

From (*) and (**):

$$N(m) = \frac{d(m)}{2}$$

as m is abitrary,

$$N(n) = \frac{d(n)}{2}$$

for all odd non-square n.

Let us compute n:

Express $n = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ where p_i are distinct prime numbers and $k_1, k_2, \ldots k_j$ are positive integers. Then if d is a divisor of n, d is of the form:

$$d = p_1^{l_1} p_2^{l_2} \cdots p_j^{l_j}$$

with $0 \leq l_i \leq k_i$ and $l_i \in \mathbb{Z}$ for all i

So that the number of possible d is the number of possible sequence $(l_1, l_2, \ldots, l_j)$ which is just:

$$(k_1 - 0 + 1)(k_2 - 0 + 1) \cdots (k_j - 0 + 1)$$

$$= (k_1 + 1)(k_2 + 1) \cdots (k_j + 1).$$

Back to problem 5 and 6!

5. $pq = p^1 q^1$

So $d(pq) = (1+1)(1+1) = 4$

and that $N(pq) = \frac{d(pq)}{2} = 2$.

6. $675 = 3^3 \times 5^2$

so $d(675) = (3+1)(2+1) = 12$

and that $N(675) = \frac{d(675)}{2} = 6$.