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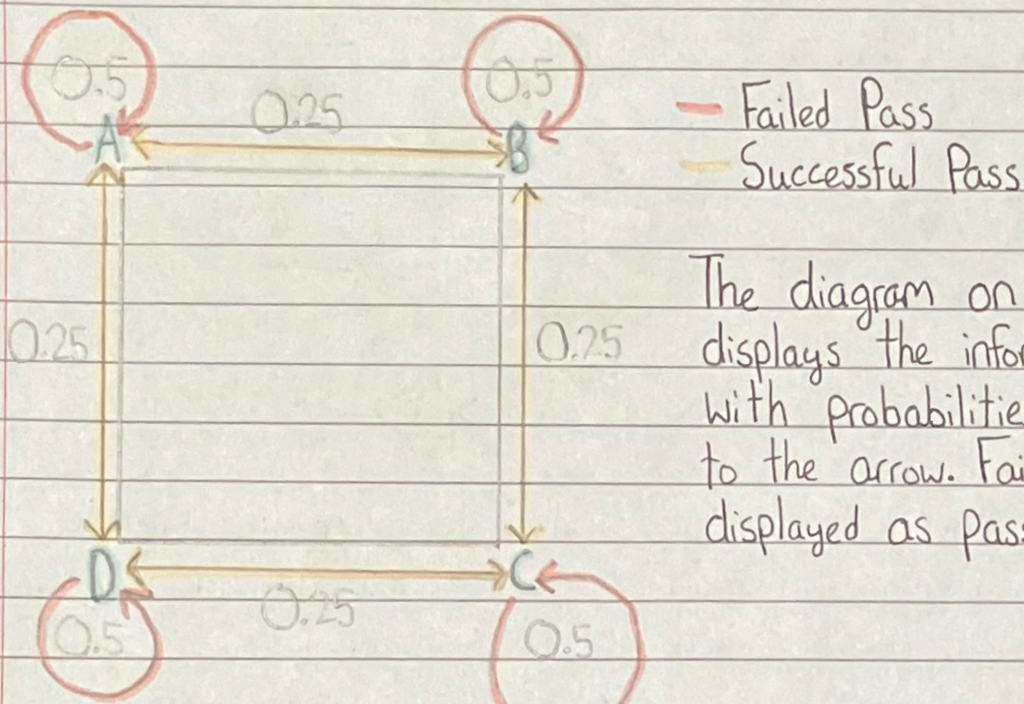
Pass the Parcel

Four children, A, B, C, and D, are playing a version of the game "pass the parcel". They stand in a circle, so that ABCDA is the clockwise order. Each time a whistle is blown, the child holding the parcel is supposed to pass the parcel immediately exactly one place clockwise.

In fact each child, independently of any other past event, passes the parcel clockwise with probability $\frac{1}{4}$, passes it anticlockwise with probability $\frac{1}{4}$ and fails to pass it at all with probability $\frac{1}{2}$.

1.



The diagram on the left displays the information above with probabilities shown next to the arrow. Failed passes are displayed as passes to oneself.

The transition matrix for this situation could be displayed as:

$$M = \begin{bmatrix} 0.5 & 0.25 & 0 & 0.25 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0.25 & 0 & 0.25 & 0.5 \end{bmatrix}$$

2. $M = \begin{bmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \end{bmatrix}$ $M^3 = \begin{bmatrix} 5/16 & 1/4 & 3/16 & 1/4 \\ 1/4 & 5/16 & 1/4 & 3/16 \\ 3/16 & 1/4 & 5/16 & 1/4 \\ 1/4 & 3/16 & 1/4 & 5/16 \end{bmatrix}$

$M^2 = \begin{bmatrix} 3/8 & 1/4 & 1/8 & 1/4 \\ 1/4 & 3/8 & 1/4 & 1/8 \\ 1/8 & 1/4 & 3/8 & 1/4 \\ 1/4 & 1/8 & 1/4 & 3/8 \end{bmatrix}$ $M^4 = \begin{bmatrix} 9/32 & 1/4 & 7/32 & 1/4 \\ 1/4 & 9/32 & 1/4 & 7/32 \\ 7/32 & 1/4 & 9/32 & 1/4 \\ 1/4 & 7/32 & 1/4 & 9/32 \end{bmatrix}$

3. As shown above, the element of the matrices can be split into three categories for all powers of M: green, red and blue.

Green: The elements in green are always $\frac{1}{4}$.

Blue: The elements in blue are always the same in each matrix.

Power of M	1	2	3	4	5	n
Value of blue elements	$\frac{2}{4}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{9}{32}$	$\frac{17}{64}$	$\frac{2^{n-1}+1}{2^{(n+1)}}$

The table above can be used to find the value of blue elements in the matrix M^n by splitting up the numerator and denominator, and finding the n^{th} term of both parts. The first term can be simplified to $\frac{1}{2}$ but is expressed as $\frac{2}{4}$, as it is easier to find the n^{th} term in this form.

Red: The elements in red are similar to the elements in blue, but the numerator is 2 less.

Value of red elements: $\frac{2^{n-1}-1}{2^{(n+1)}}$, where n is the power of M.

$$\frac{2^{(n-1)}-1}{2^{(n+1)}} = \left(\frac{2^{(n-1)}}{2^{(n+1)}} - \frac{1}{2^{(n+1)}}\right) = 2^{-2} - \left(\frac{1}{2}\right)^{(n+1)} = \frac{1}{4} - \left(\frac{1}{2}\right)^{(n+1)}$$

$$\therefore \frac{2^{(n-1)}-1}{2^{(n+1)}} = \frac{1}{4} - \left(\frac{1}{2}\right)^{(n+1)}, \quad \frac{2^{(n-1)}+1}{2^{(n+1)}} = \frac{1}{4} + \left(\frac{1}{2}\right)^{(n+1)}$$

Green: $\frac{1}{4}$

Blue $\frac{1}{4} + \left(\frac{1}{2}\right)^{n+1}$

Red: $\frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}$

$$M^n = \begin{bmatrix} \frac{1}{4}+\left(\frac{1}{2}\right)^{(n+1)} & \frac{1}{4} & \frac{1}{4}-\left(\frac{1}{2}\right)^{(n+1)} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}+\left(\frac{1}{2}\right)^{(n+1)} & \frac{1}{4} & \frac{1}{4}-\left(\frac{1}{2}\right)^{(n-1)} \\ \frac{1}{4}-\left(\frac{1}{2}\right)^{(n+1)} & \frac{1}{4} & \frac{1}{4}+\left(\frac{1}{2}\right)^{(n+1)} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}-\left(\frac{1}{2}\right)^{(n+1)} & \frac{1}{4} & \frac{1}{4}+\left(\frac{1}{2}\right)^{(n+1)} \end{bmatrix}$$

4. If the game starts with child A holding the parcel, we can use the first column of the matrix above to find the probabilities that A, B, C, or D are holding the parcel after n whistle.

Child	Probability after n whistle
A	$\frac{1}{4} + \left(\frac{1}{2}\right)^{n+1}$
B	$\frac{1}{4}$
C	$\frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}$
D	$\frac{1}{4}$

5. As n approaches infinity, the probability that child A and C are holding the parcel tends towards $\frac{1}{4}$. This is because as n increases, the exponent $\frac{1}{2}$ is raised to also increases, so becomes negligible. The probability child B and child D are holding the parcel after n whistle is always $\frac{1}{4}$.