

Markov Matrices

- a) Each day for a mid-morning snack I either have an apple or a banana. If I have an apple on one day, then the probability that I have an apple on the next day is $\frac{7}{10}$, otherwise I have a banana. If I have a banana one day then the probability that I have a banana on the next day is $\frac{4}{10}$.

Work out the probability that, if I have an apple on Monday, I have

1. An apple on Wednesday
2. A banana on Wednesday

1. If I have an apple on Monday, there is a 70% chance I have an apple on Tuesday, followed by a 70% chance on Wednesday.

$$0.7 \times 0.7 = 0.49$$

However, there is a 30% chance I have a banana on Tuesday, followed by a 60% chance I have an apple on Wednesday.

$$0.3 \times 0.6 = 0.18$$

Adding these probabilities gives us a 67% of having an apple on Wednesday.

2. If there is a 67% chance of having an apple on Wednesday, there is a 33% chance of having a banana on Wednesday.

We could also evaluate this by calculating $(0.3 \times 0.4) + (0.7 \times 0.3) = 0.33$
 $= 33\%$

b) The information could also be displayed as a matrix.

$$M = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix}$$

$$1. M^2 = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.67 & 0.66 \\ 0.33 & 0.34 \end{pmatrix}$$

The elements in the first column are the probabilities of having an apple or a banana on Wednesday, calculated previously.

$$2. M^3 = \begin{pmatrix} 0.67 & 0.66 \\ 0.33 & 0.34 \end{pmatrix} \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.667 & 0.666 \\ 0.333 & 0.334 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 0.667 & 0.666 \\ 0.333 & 0.334 \end{pmatrix} \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.6667 & 0.6666 \\ 0.3333 & 0.3334 \end{pmatrix}$$

$$\text{As } n \rightarrow \infty, M^n = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

3. On average, an apple will be chosen on $\frac{2}{3}$ of the days.

$$c) 1. \begin{pmatrix} 0 & 0.3 & 0.5 \\ 0.2 & 0.4 & 0 \\ 0.8 & 0.3 & 0.5 \end{pmatrix}$$

$$2. \begin{pmatrix} 0 & 0.3 & 0.5 \\ 0.2 & 0.4 & 0 \\ 0.8 & 0.3 & 0.5 \end{pmatrix}^4 = \begin{bmatrix} 0.3482 & 0.3111 & 0.3045 \\ 0.1004 & 0.121 & 0.107 \\ 0.5514 & 0.5679 & 0.5885 \end{bmatrix}$$

Probability of apple on Friday: 34.82%

Probability of banana on Friday: 10.04%

Probability of having cake on Friday: 55.14%

3. On average, cake will be chosen on 57.44% of days!