

Solution 1:

Method 1: [Forming simultaneous eqs to solve.]

Step 1) Check the length of the repetend.

$$\text{Let } x \sim 0.\overline{a_1 a_2 a_3} \\ \text{length} = 3$$

Step 2)

Multiply x by 10^{length}

$$\text{i.e. } x \times 10^3 = a_1 a_2 a_3.\overline{a_1 a_2 a_3}$$

Step 3)

Then see how much should be added to x to equal value found in step 2.

Step 4)

Equate eqs found in step 2 & step 3 and solve for x .

e.g) If $x = 0.8\overline{3}$

$$\text{length (repetend)} = 1$$

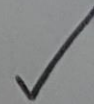
$$x \times 10^1 = 8.\overline{3}$$

$$8.\overline{3} - 0.8\overline{3} = 7.5$$

$$\text{so } x \times 10^1 = x + 7.5$$

$$9x = 7.5$$

$$x = \frac{7.5}{9}$$



Solution 2

Method 2: [generalisation from observation]

Using $S_n = \frac{a}{1-r}$, $|r| < 1$

I know that e.g. $0.\dot{4}0\dot{5}$ is
 an infinite sum of: $(0.405 + 0.000405 + 0.00000405 + \dots)$
 which can be written as $S = \frac{0.405}{1-0.901}$
 $\rightarrow S = \frac{405}{999}$ ✓

Standard Procedure

More Exotic Examples

Though I could have stopped here, I wondered whether there was an underlying pattern.
 So I tried: $1.\dot{2}3\dot{4}5$
 (I won't simplify here as I later noticed '9' is rather special)

e.g. $1.\dot{2}3\dot{4}5 \equiv 1 + 0.23 + (0.0045 + 0.000045 + \dots)$

$$= 1 + \frac{23}{100} + \frac{0.0045}{1-0.01}$$

$$= 1 + \frac{23}{100} + \frac{45}{9900}$$

$$= 1 + \frac{23(99) + 45}{9900}$$

$$= 1 + \frac{23(100-1) + 45}{9900}$$

$$= 1 + \frac{2345 - 23}{9900}$$

$\left[\frac{23}{100} \times \frac{99}{99} = \frac{23(99)}{9900} \right]$
 $= \frac{23(100-1)}{9900}$

Interesting Observation

Here, I noticed the lengths of 9s and 0s match the (highlighted) lengths of the (non-)repeats.

Further Investigation

Intrigued, I tried more egs. (namely $249.88764\dot{2}$ and $0.00234\dot{5}$)

Solution

I later noticed, in my prev. example, $1 + \frac{2345-23}{9900} \equiv \frac{1(9900) + 2345 - 23}{9900}$
 $\equiv \frac{2345 - 23}{9900}$!!!
 wow

Idea
 * After failing to find an obvious conclusion, I just decided to play around with the numbers and Voila (!), $99 = (100-1)$ & I immediately realised (inspired by my Method 1; 10 length) I was onto something.

(I soon realised that they're equivalent after hypothesising the below formula)

So if I generalise:

$$c_1c_2\dots c_l o b_1b_2\dots b_m \dot{a}_1a_2\dots a_n = \frac{(c_1c_2\dots c_l b_1b_2\dots b_m a_1a_2\dots a_n) - (c_1c_2\dots c_l b_1\dots b_m)}{99\dots 9000\dots 0}$$

(for some $l, m, n \in \mathbb{Z}^+$)

$\underbrace{\hspace{10em}}_{n \text{ 9s}}$
 $\underbrace{\hspace{10em}}_{m \text{ 0s}}$

(where $n = \text{len}(bs)$) (where $m = \text{len}(as)$)

Zoomed-in image of Solution 2

See last line highlighted

Intrigued, I tried more exs. (namely 249.887642 and 0.002345)

I later noticed, in my prev. example, $1 + \frac{2345-23}{9900} \equiv \frac{2(9900) + 2345-23}{9900}$

(I immediately realised (inspired by my Method 1) to length) I was onto something.

(I soon realised that they're equivalent after hypothesising the below formula)

So if I generalise:

$$\underbrace{c_1 c_2 \dots c_l}_l \underbrace{b_1 b_2 \dots b_m}_m \underbrace{a_1 a_2 \dots a_n}_n = \frac{(c_1 c_2 \dots c_l b_1 b_2 \dots b_m a_1 a_2 \dots a_n) - (c_1 c_2 \dots c_l b_1 \dots b_m)}{999\dots9 \underbrace{000\dots0}_m}$$

(for some $l, m, n \in \mathbb{Z}^+$)

(where $n = \text{len}(b_s)$) (where $m = \text{len}(a_s)$)