Solution 1:

Method 1 : [Forming simultaneous eqs to solve]

Step 1) Check the length of the repetend.

$\boxed{\text{Let } x \sim 0.\underbrace{\dot{a}_1 a_2 \dot{a}_3}_{\text{length} = 3}}$

Step 2) Multiply x by 10^{length}

i.e. $x \times 10^3 = a_1 a_2 a_3 . \dot{a}_1 a_2 \dot{a}_3$

Step 3) Then see how much should be added to x to equal value found in step 2.

Step 4) Equate eqs formed in step 2 & step 3 and solve for x.

e.g.) If $x = 0.8\dot{3}$

$\text{length}(x \text{ repetend}) = 1$

$x \times 10^1 = 8.\dot{3}$

$8.\dot{3} - 0.8\dot{3} = 7.5$

so $x \times 10^1 = x + 7.5$

$9x = 7.5$

$x = \frac{75}{90}$ ✓

Solution 2

Method 2: [generalisation from observation]

Using $S_\infty = \dfrac{a}{1-r}$, $|r|<1$

Standard Procedure

- I know ~~that~~ e.g. $0.\dot{4}0\dot{5}$ is an infinite sum of: $(0.405 + 0.000405 + 0.000000405 + \cdots)$

  which can be written as $S = \dfrac{0.405}{1-0.001}$

  $\Rightarrow S = \dfrac{405}{999}$ ✓

  (I won't simplify here as I later noticed '9' is rather special)

More exotic Examples

- Though I could have stopped here, I wondered whether there was an underlying pattern.
- So I tried: $\boxed{1.23\dot{4}\dot{5}}$

e.g.

$$
\begin{aligned}
1.23\dot{4}\dot{5} &\equiv 1 + 0.23 + (0.0045 + 0.000045 + \cdots)\\
&= 1 + \frac{23}{100} + \frac{0.0045}{1-0.01}\\
&= 1 + \frac{23}{100} + \frac{45}{9900}\\
&= 1 + \frac{23(99)+45}{9900}\\
&= 1 + \frac{23(100-1)+45}{9900}\\
&= 1 + \frac{2345-23}{9900}
\end{aligned}
$$

$$\left(\frac{23}{100}\times\frac{99}{99} = \frac{23(99)}{9900} = \frac{23(100-1)}{9900}\right)$$

Interesting Observation

Idea

★ After failing to ~~find an obvious~~ conclusion, I just decided to play around with the numbers and Voila (!), $99 = (100-1)$ & I immediately realised (inspired by my Method 1; 10^{length}) I was onto something.

- Here, I noticed the lengths of 9s and 0s match the (highlighted) lengths of the (non-/) repetends.

Further Investigation

- Intrigued, I tried more egs. (namely $249.88\dot{7}64\dot{2}$ and $0.002\dot{3}4\dot{5}$)

Solution

- I later noticed, in my prev. example, $1 + \dfrac{2345-23}{9900} \equiv \dfrac{1(9900)+2345-23}{9900} \equiv \dfrac{12345-123}{9900}$!!! wow

  (I soon realised that they're equivalent after hypothesising the below formula)

So if I generalise:

$$\underline{c_1c_2\cdots c_l}\,.\,\underline{b_1b_2\cdots b_m}\,\underline{\dot{a}_1a_2\cdots\dot{a}_n} = \frac{(c_1c_2\cdots c_l b_1b_2\cdots b_m a_1a_2\cdots a_n) - (c_1\cdots c_l b_1\cdots b_m)}{\underbrace{99\cdots9}_{n\ 9s}\underbrace{00\cdots0}_{m\ 0s}}$$

(for some $l, m, n \in \mathbb{Z}^+$)

(where $n = \text{len}(b_s)$) (where $m = \text{len}(a_s)$)

<u>Zoomed-in image of Solution 2</u>

See last line highlighted

Intrigued, I tried more egs. (namely $249.88\dot{7}64\dot{2}$ and $0.002\dot{3}4\dot{5}$)

& I immediately realised (inspired by my Method I & 10^{length}) I was onto something.

I later noticed, in my prev. example, $1 + \frac{2345-23}{9900} \equiv \frac{1(9900)+2345-23}{9900}$

!!! wow

(I soon realised that they're equivalent after hypothesising the below formula)

So if I generalise:

$$c_1c_2\cdots c_l.b_1b_2\cdots b_m\dot{a}_1a_2\cdots\dot{a}_n = \frac{(c_1c_2\cdots c_l b_1b_2\cdots b_m a_1a_2\cdots a_n)-(c_1\cdots c_l b_1\cdots b_m)}{\underbrace{999\cdots9}_{n\ 9\text{s}}\underbrace{000\cdots0}_{m\ 0\text{s}}}$$

(for some $l, m, n \in \mathbb{Z}^+$)

(where $n = \text{len}(b_s)$) (where $m = \text{len}(a_s)$)