Pythagoras for a tetrahedron

Since the question asks about areas, it makes sense to work out the areas of some of the simple right-angle triangles. So for area R we have:

base = a, height = c $\therefore$ $R = \frac{ac}{2}$
then we do the same for Q and P to reach
$Q = \frac{bc}{2}$ and $P = \frac{ab}{2}$

Now all we need is area S. This is not a right angle triangle so we know we will need to use $\frac{1}{2}ab\sin C$ to find area, meaning we will need sides and angles of this ~~triangle~~ CAB. The only way to find angles will be with cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$ so we will need all the sides of this triangle, which we can see can be found using pythagoras:

$$(AB) = \sqrt{(AO)^2 + (AB)^2} = \sqrt{a^2 + b^2}$$
$$(AC) = \sqrt{a^2 + c^2}$$
$$(CB) = \sqrt{b^2 + c^2}$$

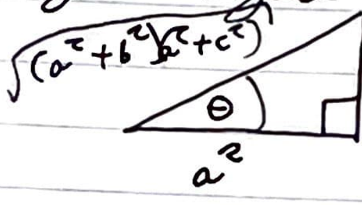
Now using cosine rule we have: (using A = angle at A etc):

$$\sqrt{b^2+c^2}^2 = \sqrt{a^2+b^2}^2 + \sqrt{a^2+c^2}^2 - 2\sqrt{a^2+b^2}\sqrt{a^2+c^2}\cos A$$
$$b^2 + c^2 = 2a^2 + b^2 + c^2 - 2\sqrt{(a^2+b^2)(a^2+c^2)}\cos A$$
$$2\sqrt{(a^2+b^2)(a^2+c^2)}\cos A = 2a^2$$
$$\cos A = \frac{a^2}{\sqrt{(a^2+b^2)(a^2+c^2)}}$$
$$A = \cos^{-1}\left(\frac{a^2}{\sqrt{(a^2+b^2)(a^2+c^2)}}\right)$$

so we have an expression for angle CAB but we want sin of this angle ~~and~~ so

$$\sin A = \sin\left(\cos^{-1}\left(\frac{a^2}{\sqrt{(a^2+b^2)(a^2+c^2)}}\right)\right)$$

so how do we evaluate this $\sin(\cos^{-1}(x))$. we can think of what $\cos^{-1}(x)$ equals using the formula $\cos\theta = A/H$ so $\theta = \cos^{-1}(A/H)$ and we can represent this on a right-angled triangle: with $a^2 = A$ and $H = \sqrt{(a^2+b^2)(a^2+c^2)}$



and we want $\sin\theta$ which we know is O/H so we need to find the opposite via pythagoras:

$$\sqrt{(a^2+b^2)(a^2+c^2)}^2 - (a^2)^2 = O^2$$

$$a^4 + a^2b^2 + a^2c^2 + b^2c^2 - a^4 = O^2 = a^2b^2 + a^2c^2 + b^2c^2$$

so $\sin\theta = \dfrac{\cancel{a}\sqrt{a^2b^2+a^2c^2+b^2c^2}}{\sqrt{(a^2+b^2)(a^2+c^2)}}$ now using sine rule we have

$$\frac{1}{2} \times \sqrt{a^2+b^2} \times \sqrt{a^2+c^2} \times \frac{\sqrt{a^2b^2+a^2c^2+b^2c^2}}{\sqrt{(a^2+b^2)(a^2+c^2)}}$$

$$\equiv \frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2} = S$$

so ~~Subst~~ substituting all into $P^2+Q^2+R^2=S^2$ we get

$$\left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ac}{2}\right)^2 = \left(\frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}\right)^2$$

$$\frac{a^2b^2}{4} + \frac{b^2c^2}{4} + \frac{a^2c^2}{4} = \frac{1}{4}(a^2b^2+a^2c^2+b^2c^2)$$

which are the same so $P^2+Q^2+R^2=S^2$ is always true

□.