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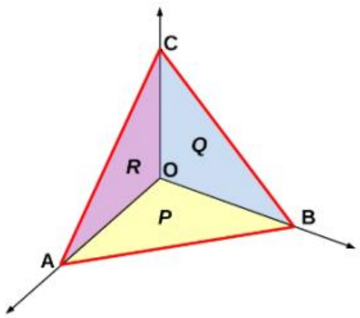
Tuesday, November 8, 2022 6:06 PM

Pythagoras for a Tetrahedron

Age 16 to 18
Challenge Level ★★★



Consider a right-angled tetrahedron with vertices at $O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$.

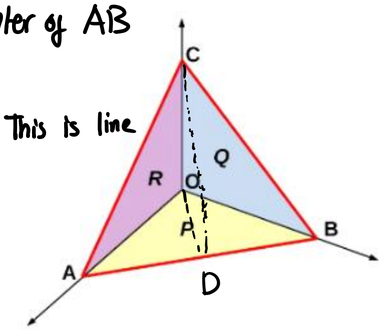
Let the area of face AOB be P, the area of BOC be Q and the area of COA be R. Also let the slanted face ABC have area S.
(*S is not shown on the diagram above!*).

Can you prove that $P^2+Q^2+R^2=S^2$?

Equivalently: (area OBC)^2 +(area OCA)^2 +(area OAB)^2 =(area ABC)^2.

Lets draw a line that reaches the center of AB from point C. We will call this line CD

Lets also draw a line from point D to O. This is line DO.



$P=\left(\frac{1}{2}\times AO\times BO\right)$ or $\left(\frac{1}{2}\times AB\times DO\right)$

$Q=\left(\frac{1}{2}\times BO\times CO\right)$

$R=\left(\frac{1}{2}\times AO\times CO\right)$

$S=\left(\frac{1}{2}\times AB\times DO\right)$

$\frac{1}{2}\times AO\times BO=\frac{1}{2}\times AB\times DO$

Lets square both sides

$\frac{1}{4}\times AO^2\times BO^2=\frac{1}{4}\times AB^2\times DO^2$

If we join the points C, D and O, we get a right-angled triangle

According to the Pythagorean theorem,

① $DO^2=CD^2-CO^2$

$\frac{1}{4}\times AO^2\times BO^2=\frac{1}{4}\times AB^2\times DO^2$

Lets substitute ① into DO^2

$\frac{1}{4}\times AO^2\times BO^2=\frac{1}{4}\times AB^2\times(CD^2-CO^2)$

$\frac{1}{4}\times AO^2\times BO^2=\left(\frac{1}{4}\times AB^2\times CD^2\right)-\left(\frac{1}{4}\times AB^2\times CO^2\right)$

Lets look at triangle ABO

According to the Pythagorean theorem, we find:

② $AB^2=AO^2+BO^2$

$\frac{1}{4}\times AO^2\times BO^2=\left(\frac{1}{4}\times AB^2\times CD^2\right)-\left(\frac{1}{4}\times AB^2\times CO^2\right)$

$\left(\frac{1}{4}\times AO^2\times BO^2\right)+\left(\frac{1}{4}\times AB^2\times CO^2\right)=\left(\frac{1}{4}\times AB^2\times CD^2\right)$

By substitute ② in AB^2

$$\left(\frac{1}{4}\times AO^2\times BO^2\right)+\left(\frac{1}{4}\times (AO^2+BO^2)\times CO^2\right)=\frac{1}{4}\times AB^2\times CD^2$$

$$\left(\frac{1}{4}\times AO^2\times BO^2\right)+\left(\frac{1}{4}\times AO^2\times CO^2\right)+\left(\frac{1}{4}\times BO^2\times CO^2\right)=\frac{1}{4}\times AB^2\times CD^2$$

Lets simplify this equation

$$\left(\frac{1}{2}\times AO\times BO\right)^2+\left(\frac{1}{2}\times AO\times CO\right)^2+\left(\frac{1}{2}\times BO\times CO\right)^2=\left(\frac{1}{2}\times AB\times CD\right)^2$$

$$\left.\begin{aligned} P&=\frac{1}{2}\times AO\times BO\\ Q&=\frac{1}{2}\times BO\times CO\\ R&=\frac{1}{2}\times AO\times CO\\ S&=\frac{1}{2}\times AB\times CD \end{aligned}\right\} ③$$

By substituting from ③

$$P^2+R^2+Q^2=S^2$$

$$\Rightarrow \underline{P^2+Q^2+R^2=S^2}$$