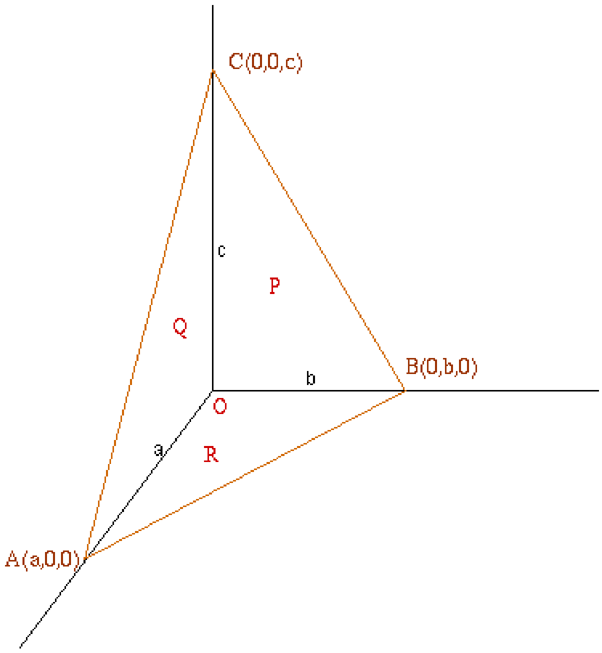
Consider a tetrahedron *ABCO* with three right angles at the point *O* (diagram looking down on *O*).
Prove that the sum of the squares of the areas marked *P*, *Q* and *R* is equal to the square of the area of the triangle *ABC*.



Using the formula for the area of a triangle $\frac{1}{2}bh$:

(for notational convenience let Δ represent the area of the triangle formed by connecting the three points which follow)

$$\Delta OAB = \frac{1}{2}ab \qquad (\Delta OAB)^2 = \frac{1}{4}a^2b^2$$
$$\Delta OBC = \frac{1}{2}bc \quad \text{and so} \quad (\Delta OBC)^2 = \frac{1}{4}b^2c^2$$
$$\Delta OCA = \frac{1}{2}ca \qquad (\Delta OCA)^2 = \frac{1}{4}c^2a^2$$

Therefore the sum of the squared areas *P*, *Q* and *R* is:
$$(\Delta OAB)^2+(\Delta OBC)^2+(\Delta OCA)^2 = \frac{1}{4}(a^2b^2+b^2c^2+c^2a^2)$$

It is also apparent due to Pythagoras' theorem that: (as the triangles *OAB*, *OBC* and *OCA* contain right angles at the point *O*)
$$AB = \sqrt{a^2+b^2}$$
$$BC = \sqrt{b^2+c^2}$$
$$CA = \sqrt{c^2+a^2}$$

Using the cosine rule then simplifying:
$$\cos(\sphericalangle CAB) = \frac{(\sqrt{a^2+b^2})^2+(\sqrt{a^2+c^2})^2-(\sqrt{b^2+c^2})^2}{2\sqrt{(a^2+b^2)(a^2+c^2)}}$$

$$\sphericalangle CAB = \cos^{-1}\left(\frac{(\sqrt{a^2+b^2})^2+(\sqrt{a^2+c^2})^2-(\sqrt{b^2+c^2})^2}{2\sqrt{(a^2+b^2)(a^2+c^2)}}\right)$$

$$\sphericalangle CAB = \cos^{-1}\left(\frac{a^2+b^2+a^2+c^2-(b^2+c^2)}{2\sqrt{(a^2+b^2)(a^2+c^2)}}\right)$$

$$\sphericalangle CAB = \cos^{-1}\left(\frac{2a^2}{2\sqrt{(a^2+b^2)(a^2+c^2)}}\right)$$

$$\sphericalangle CAB = \cos^{-1}\left(\frac{a^2}{\sqrt{(a^2+b^2)(a^2+c^2)}}\right)$$

Now the formula for the area of a triangle in terms of two of it's edges and the angle inbetween is:

$\frac{1}{2}ab\sin(C)$ (where C is the angle and a and b are the edges either side)

However the relationship between sine and cosine is such that:

$$(\sin(\theta))^2+(\cos(\theta))^2 = 1$$
$$\sin(\theta) = \sqrt{1-(\cos(\theta))^2}$$

The formula for the area of a triangle may be written as:

$$\frac{1}{2}ab\sqrt{1-(\cos(C))^2}$$

Substituting in $a = CA$, $b = AB$, $C = \measuredangle CAB$ then simplifying:

$$\Delta ABC = \frac{1}{2}\sqrt{c^2+a^2}\sqrt{a^2+b^2}\sqrt{1-(\cos(\cos^{-1}(\frac{a^2}{\sqrt{(a^2+b^2)(a^2+c^2)}})))^2}$$

$$\Delta ABC = \frac{1}{2}\sqrt{c^2+a^2}\sqrt{a^2+b^2}\sqrt{1-\frac{a^4}{(a^2+b^2)(a^2+c^2)}}$$

$$\Delta ABC = \frac{1}{2}\sqrt{(c^2+a^2)(a^2+b^2)(1-\frac{a^4}{(a^2+b^2)(a^2+c^2)})}$$

$$\Delta ABC = \frac{1}{2}\sqrt{(c^2+a^2)(a^2+b^2)(\frac{(a^2+b^2)(a^2+c^2)-a^4}{(a^2+b^2)(a^2+c^2)})}$$

$$\Delta ABC = \frac{1}{2}\sqrt{(a^2+b^2)(a^2+c^2)-a^4}$$

$$\Delta ABC = \frac{1}{2}\sqrt{a^4+a^2c^2+b^2a^2+b^2c^2-a^4}$$

$$\Delta ABC = \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

Squaring this:

$$\boldsymbol{(\Delta ABC)^2 = \frac{1}{4}(a^2b^2+b^2c^2+c^2a^2) = (\Delta OAB)^2+(\Delta OBC)^2+(\Delta OCA)^2}$$

Therefore the sum of the squares of the areas of the three faces around point *O* is equal to the square of the area of the face opposite the point *O*.