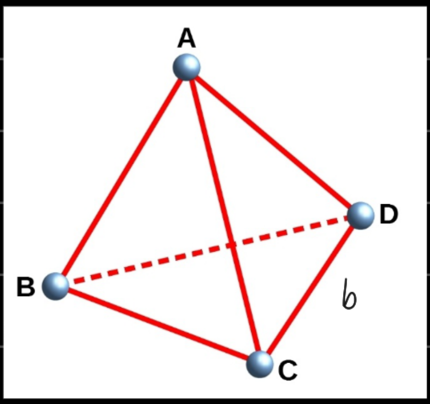
Q. Show AD and BC are mutually perpendicular if and only if $AB^2 + CD^2 = AC^2 + BD^2$.

First, we can recall the geometric definition of the dot/scalar product:

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$, where $\underline{a}$ and $\underline{b}$ are two vectors and θ is the angle between them.

If $\underline{a} = \underline{b}$, then we get $\underline{a} \cdot \underline{a} = |\underline{a}||\underline{a}|\cos 0$ — the angle between a line and itself is 0.



$\Rightarrow \underline{a} \cdot \underline{a} = |\underline{a}|^2$.

this is equivalent to saying $AB^2 + CD^2 = AC^2 + BD^2$.

If $AB^2 + CD^2 - AC^2 - BD^2 = 0$, and we lable the position vectors of A, B, C and D to be $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ respectively, we get:

$$LHS = (\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a}) + (\underline{d}-\underline{c})\cdot(\underline{d}-\underline{c}) - (\underline{c}-\underline{a})\cdot(\underline{c}-\underline{a}) - (\underline{d}-\underline{b})\cdot(\underline{d}-\underline{b})$$

We can use the distributive property of the scalar product to obtain the following result:

$$= \cancel{\underline{b}\cdot\underline{b}} - \underline{a}\cdot\underline{b} - \underline{a}\cdot\underline{b} + \cancel{\underline{a}\cdot\underline{a}} + \cancel{\underline{d}\cdot\underline{d}} - \underline{c}\cdot\underline{d} - \underline{c}\cdot\underline{d} + \cancel{\underline{c}\cdot\underline{c}} - \cancel{\underline{c}\cdot\underline{c}} + \underline{a}\cdot\underline{c} + \underline{a}\cdot\underline{c} - \cancel{\underline{a}\cdot\underline{a}}$$
$$- \cancel{\underline{d}\cdot\underline{d}} + \underline{b}\cdot\underline{d} + \underline{b}\cdot\underline{d} - \cancel{\underline{b}\cdot\underline{b}}$$

$$= -2(\underline{c}\cdot\underline{d} - \underline{b}\cdot\underline{d} - \underline{a}\cdot\underline{c} + \underline{a}\cdot\underline{b})$$

$$= -2\big((\underline{d}-\underline{a})\cdot(\underline{c}-\underline{b})\big)$$

Therefore, $AB^2 + CD^2 = AC^2 + BD^2$ <u>if and only if</u> $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{BC}$.
(using the fact that two vectors and perpendicular <u>if and only if</u> their dot product is 0)