

Firstly, we see that when we multiply matrices :

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} Aa + Bd + Cg & Ab + Be + Ch & Ac + Bf + Ci \\ Da + Ed + Fg & Db + Ee + Fh & Dc + Ef + Fi \\ Ga + Hd + Ig & Gb + He + Ih & Gc + Hf + Ii \end{pmatrix}$$

And so on for larger matrices.

Hence when the matrix gives probabilities :

$$\mathbf{A} = \begin{pmatrix} P(1 \rightarrow 1) & P(1 \rightarrow 2) & P(1 \rightarrow 3) & P(1 \rightarrow 4) \\ P(2 \rightarrow 1) & P(2 \rightarrow 2) & P(2 \rightarrow 3) & P(2 \rightarrow 4) \\ P(3 \rightarrow 1) & P(3 \rightarrow 2) & P(3 \rightarrow 3) & P(3 \rightarrow 4) \\ P(4 \rightarrow 1) & P(4 \rightarrow 2) & P(4 \rightarrow 3) & P(4 \rightarrow 4) \end{pmatrix}$$

Where $P(1 \rightarrow 2)$ is the probability of travelling to 2 when at 1

Then the first column of \mathbf{A}^2 is

$$\begin{pmatrix} P(1 \rightarrow 1) \cdot P(1 \rightarrow 1) + P(1 \rightarrow 2) \cdot P(2 \rightarrow 1) + P(1 \rightarrow 3) \cdot P(3 \rightarrow 1) + P(1 \rightarrow 4) \cdot P(4 \rightarrow 1) \\ P(2 \rightarrow 1) \cdot P(1 \rightarrow 1) + P(2 \rightarrow 2) \cdot P(2 \rightarrow 1) + P(2 \rightarrow 3) \cdot P(3 \rightarrow 1) + P(2 \rightarrow 4) \cdot P(4 \rightarrow 1) \\ P(3 \rightarrow 1) \cdot P(1 \rightarrow 1) + P(3 \rightarrow 2) \cdot P(2 \rightarrow 1) + P(3 \rightarrow 3) \cdot P(3 \rightarrow 1) + P(3 \rightarrow 4) \cdot P(4 \rightarrow 1) \\ P(4 \rightarrow 1) \cdot P(1 \rightarrow 1) + P(4 \rightarrow 2) \cdot P(2 \rightarrow 1) + P(4 \rightarrow 3) \cdot P(3 \rightarrow 1) + P(4 \rightarrow 4) \cdot P(4 \rightarrow 1) \end{pmatrix}$$

And so on for four columns.

Now, the probability of getting from 1 to 2 in x steps equals the sum the probabilities of each way of getting from 1 to 2 in x steps :

$$\begin{aligned} &P(1 \rightarrow 2(2 \text{ steps})) \\ &= P(1 \rightarrow 1) \cdot P(1 \rightarrow 2) + P(1 \rightarrow 2) \cdot P(2 \rightarrow 2) + P(1 \rightarrow 3) \cdot P(3 \rightarrow 2) + \dots \end{aligned}$$

Thus we have in \mathbf{A}^2 the probability of getting from one point to another in two steps, as each entry in the matrix is the sum of probabilities of getting between the two points in two steps.

Similarly, we find that if we cube the matrix, we are adding the possibilities of three – stage routes between two points, and so on for higher powers.

Now, we look at what happens for higher powers of \mathbf{A} .

$$\mathbf{A}^{20} = \begin{pmatrix} 0 & 0 & 0.008 & 0.992 \\ 0.008 & 0 & 0 & 0.992 \\ 0 & 0.016 & 0 & 0.984 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{21} = \begin{pmatrix} 0.008 & 0 & 0 & 0.992 \\ 0 & 0.008 & 0 & 0.992 \\ 0 & 0 & 0.008 & 0.984 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{22} = \begin{pmatrix} 0 & 0.008 & 0 & 0.992 \\ 0 & 0 & 0.004 & 0.996 \\ 0.008 & 0 & 0 & 0.992 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

First (and easiest) we explain the fourth row. In the network, an object at (4) has a probability of 1 of returning to (4) at the next journey. It therefore always returns to (4) and never reaches any other vertex.

Next, the cycling of the three values in the upper left. At (3) and (1), an object may only travel to one other vertex ((1) and (2) respectively), and at (2) travels to (3) (with a probability of 0.5 which we ignore to explain the cycling). Therefore, an object at (1) will travel to (2), then (3,) then (1) and so on. However, it moves from one vertex at time, so it cycles through the vertices : in n journeys, the object will be at vertex $n(\bmod 3)$. Similarly, if it starts at (2), it will be at vertex $(n + 1)(\bmod 3)$ and, from (3), at $(n + 2)(\bmod 3)$.

Lastly, the numerical values of the probabilities. Every time an object is at (2), it travels to (4) with a probability p where $p = 0.5$. Therefore, it will only avoid going to (4) with a probability of $(1 - p)^m$, where m is the number of times an object is at (2). ($m \approx \frac{3}{n}$). Therefore, the probability of an object starting at (1), (2) or (3) arriving at (4) is $1 - (1 - p)^m$. As once the object is at (4) it cannot travel to any other vertex, the probability of ever being at (4) in n journeys can only increase.

With this information, we can predict the behaviour of \mathbf{A}^n as we let n increase.

The fourth row will remain at 0 0 0 1 as an object at (4) will remain at (4).

All values in the fourth column will tend to one as n tends to infinity, as they are given (roughly) by $1 - 0.5^{\frac{n}{3}}$ (as $0.5^{\frac{n}{3}}$ tends to 0).

The other nine values will cycle as shown above, in the manner :

$$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{a}{2} \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}, \begin{pmatrix} \frac{a}{2} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{c}{2} \end{pmatrix}$$

and so on for $n = 3x + 1$, $3x + 2$ and $3x + 3$ respectively (where x is

an integer).

The values of a , b and c will tend to zero, as they are given by 0.5^m .

$m = (\text{floor})(\frac{n+s}{3})$ for $s = 1, 2$ and $m = (\text{floor})(\frac{n}{3})$ for $s = 3$ where s is the number of the start vertex and (floor) is the largest integer below a number.

Hence

$$\mathbf{A}^{100} = \begin{pmatrix} 0 & 1 \times 10^{-10} & 0 & 0.9999999999 \\ 0 & 0 & 5 \times 10^{-11} & 0.9999999995 \\ 1 \times 10^{-10} & 0 & 0 & 0.9999999999 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$