Stars

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My initial observations for this beautiful problem intertwining Geometry and Number theory was that when you choose a step size that isn't a factor of the number of points chosen, then we might achieve the result of striking all points.

But soon, with multiple trials, I found out that for example, even though 4 is not a factor of 10, all points are not visited!?

Something was off, but I couldn't figure out why..
My initial claim and observation revolved around this idea that if the step size IS a factor of the number of points, after just the first round, it will return back to the starting point and loop back over and over. Hence obviously missing out a few points. Selecting 2 and 5 as the step size on 10 points show that many points are not visited.

The second step led me to discover that the steps and the number of points should be co-prime. It would be better to define step size and number of points in an ordered pair for convenience. So, the example of (4,10) allowed me to see that the line drawn for the star does not visit the starting point in the first round, but in the second round. This hinted at the fact that 4 x 5 = 10 x 2. A correlation could be seen between the number of lines drawn in the star (5) and the round number which is (2) in this specific case.

My claim is thus intuitively, "Step size" x "Lines drawn" = "Points" x "No. of rounds".

For a complete star to be drawn with all points visited, the number of lines drawn must be equal to the number of points chosen. Thus, the step size must equal the number of rounds. In the case of (4,10), this is not seen, and thus all lines are not covered.

To ensure that a complete star is drawn, the logical way is to select the step size as one that is co-prime with the number of points. That way, the step size has to correspond to the all factors of the no of rounds, essentially making “step size” = “no of rounds”. And voila, the “Lines drawn” is now equal to the “number of point” fulfilling the condition of making a star that visits each point.

After taking a look at the Euler's Totient Function, we can generalise it to say that the possible step sizes are $\varphi(n)$ where n is the number of points chosen.
From multiple trials, I even verified that the two match perfectly.

n	φ(n)
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4
11	10
12	4
13	12
14	6
15	8
16	8
17	16
18	6
19	18
20	8
21	12
22	10
23	22
24	8
25	20
26	12
27	18
28	12
29	28
30	8
31	30
32	16
33	20
34	16
35	24
36	12
37	36
38	18
39	24
40	16

For points that are prime number, all step sizes work! Another application of Euler's totient function that states that $\varphi(p) = p - 1$, where p is a prime number.