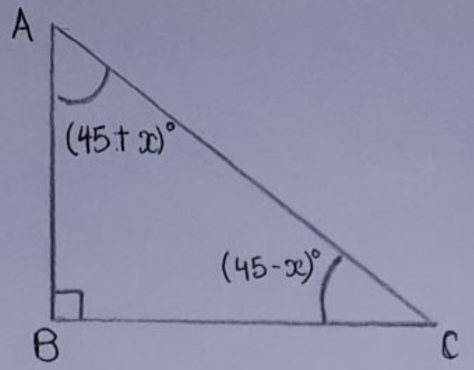
Q. Sketch a triangle with angles 90°, $(45+x)^\circ$ and $(45-x)^\circ$. Use your diagram, and Pythagoras Theorem to help you find the sum:

$$\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 89^\circ + \sin^2 90^\circ$$

Find also:

$$\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 359^\circ + \sin^2 360^\circ$$

Solution:



$\cos(45+x)^\circ = \dfrac{AB}{BC}$, $\sin(45-x)^\circ = \dfrac{AB}{AC}$, $\cos(45-x)^\circ = \dfrac{BC}{AC}$, $\sin(45+x)^\circ = \dfrac{BC}{AC}$

$\therefore \sin(45-x)^\circ = \cos(45+x)^\circ$, $\sin(45+x)^\circ = \cos(45-x)^\circ$

$\Rightarrow \sin^2(45-x)^\circ = \cos^2(45+x)^\circ$, $\sin^2(45+x)^\circ = \cos^2(45-x)^\circ$ ①

Using ①,

when $x = 45^\circ$

$\sin^2(0)^\circ = \cos^2(90)^\circ$,

when $x = 0$, $\sin^2(45)^\circ = \cos^2(45)^\circ$

when $x = 44$, $\sin^2(89)^\circ = \cos^2(1)^\circ$

when $x = 43$, $\sin^2(88)^\circ = \cos^2(2)^\circ$

and so on.

Substituting these values in

$$\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 89^\circ + \sin^2 90^\circ$$

we get

$$\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 45^\circ + \dots + \cos^2 2^\circ + \cos^2 1^\circ + \sin^2 90^\circ \quad ②$$

Since $\sin^2 x + \cos^2 x = 1$

Eq. 2 becomes, $44 \times 1 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = 45 + \frac{1}{2} = \frac{91}{2} = \underline{\underline{45.5}}$

$\rightarrow \sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 359^\circ + \sin^2 360^\circ$ — ③

$\sin(359)^\circ = \sin(-1)^\circ = -\sin(1)^\circ$

$\sin^2(359)^\circ = -\sin(1)^\circ \times -\sin(1)^\circ = \sin^2(1)^\circ$

Similarly, $\sin^2(2)^\circ = \sin^2(358)^\circ$ ... and so on.

Substituting these values in eq ③, we get,

$2(\sin^2(1)^\circ + \sin^2(2)^\circ + \dots + \sin^2(178)^\circ + \sin^2(179)^\circ) + \sin^2(180)^\circ + \sin^2(360)^\circ$.

Since, $\sin^2(180)^\circ = 0 = \sin^2(360)^\circ$,

the equation becomes,

$2(\sin^2(1)^\circ + \sin^2(2)^\circ + \dots + \sin^2(178)^\circ + \sin^2(179)^\circ)$ — ④

$\Rightarrow \sin(180 - x) = \sin 180 \cos x - \sin x \cos 180 = \sin x$

$[\sin(A-B) = \sin A \cos B - \cos A \sin B]$ $[\cos 180 = -1]$

So, $\sin(179)^\circ = \sin(180-1)^\circ = \sin(1)^\circ$

$\sin(178)^\circ = \sin(2)^\circ$

and so on.

$\therefore$ the eq. ④ becomes,

$2(2(\sin^2 1^\circ + \sin^2 2^\circ + \dots + \sin^2 89^\circ) + \sin^2 90^\circ)$ — ⑤

We know that $\sin^2(1)^\circ + \sin^2(2)^\circ + \dots + \sin^2(89)^\circ = 45.5 - \sin^2(90)^\circ = 44.5$.

Substituting it in equation ⑤, we get

$2(2(44.5)+1) = 2(89+1) = 2 \times 90 = \underline{\underline{180}}$

SOLVED BY

GAUTHAM KRISHNA.K