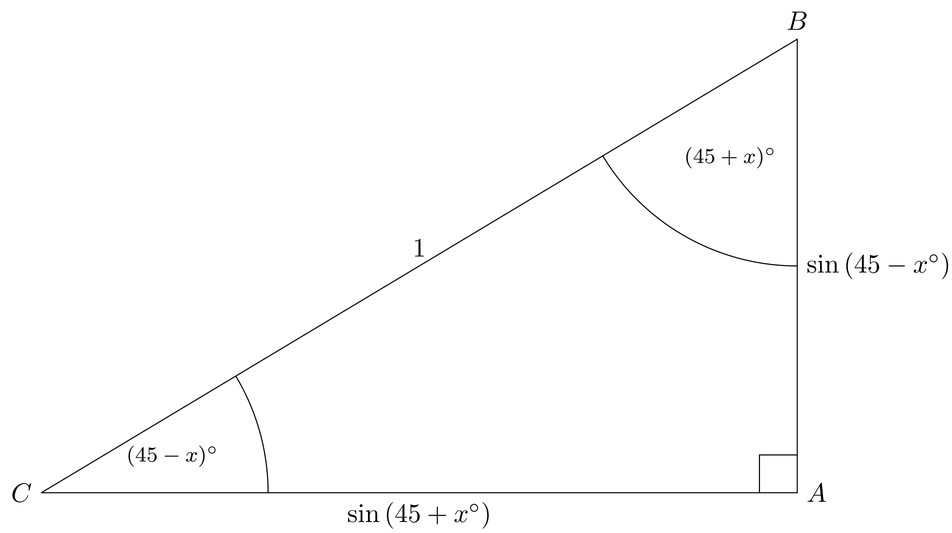
Consider the diagram below.



Using Pythagoras' theorem on the triangle above, we have

$$\sin^2(45-x^\circ)+\sin^2(45+x^\circ)=1.$$

We can evaluate our sum

$$\sin^2(1^\circ)+\sin^2(2^\circ)+\cdots+\sin^2(89^\circ)+\sin^2(90^\circ)$$

as follows.

$$\begin{aligned}
&\sin^2(1^\circ)+\sin^2(2^\circ)+\cdots+\sin^2(89^\circ)+\sin^2(90^\circ)\\
&=\sum_{x=1}^{44}\left[\sin^2(45-x^\circ)+\sin^2(45+x^\circ)\right]+\sin^2(45^\circ)+\sin^2(90^\circ)\\
&=\sum_{x=1}^{44}1+\sin^2(45^\circ)+\sin^2(90^\circ)\\
&=44+\left(\frac{\sqrt{2}}{2}\right)^2+1^2\\
&=\frac{91}{2}
\end{aligned}$$

Similarly, we can use the identity

$$\sin^2(180-\theta) \equiv \sin^2(360-\theta) \equiv \sin^2(\theta)$$

to evaluate the sum

$$\sin^2(1^\circ) + \sin^2(2^\circ) + \cdots + \sin^2(359^\circ) + \sin^2(360^\circ).$$

We know that

$$\begin{aligned}
\sin^2(45-x^\circ) + \sin^2(45+x^\circ) &\equiv \sin^2(135-x^\circ) + \sin^2(135+x^\circ) \\
&\equiv \sin^2(225-x^\circ) + \sin^2(225+x^\circ) \\
&\equiv \sin^2(315-x^\circ) + \sin^2(315+x^\circ) \equiv 1
\end{aligned}$$

from repeated use of the above identities, so

$$\begin{aligned}
\sum_{i=1}^{360} \sin^2(i^\circ) = &\left(\sum_{x=1}^{44}\left[\sin^2(45-x^\circ) + \sin^2(45+x^\circ)\right] + \sin^2(45^\circ) + \sin^2(90^\circ)\right) \\
&+ \left(\sum_{x=1}^{44}\left[\sin^2(135-x^\circ) + \sin^2(135+x^\circ)\right] + \sin^2(135^\circ) + \sin^2(180^\circ)\right) \\
&+ \left(\sum_{x=1}^{44}\left[\sin^2(225-x^\circ) + \sin^2(225+x^\circ)\right] + \sin^2(225^\circ) + \sin^2(270^\circ)\right) \\
&+ \left(\sum_{x=1}^{44}\left[\sin^2(315-x^\circ) + \sin^2(315+x^\circ)\right] + \sin^2(315^\circ) + \sin^2(360^\circ)\right) \\
= &\left(\sum_{x=1}^{44} 1 + \sin^2(45^\circ) + \sin^2(90^\circ)\right) + \left(\sum_{x=1}^{44} 1 + \sin^2(135^\circ) + \sin^2(180^\circ)\right) \\
&+ \left(\sum_{x=1}^{44} 1 + \sin^2(225^\circ) + \sin^2(270^\circ)\right) + \left(\sum_{x=1}^{44} 1 + \sin^2(315^\circ) + \sin^2(360^\circ)\right) \\
= &\left(44 + \sin^2(45^\circ) + \sin^2(90^\circ)\right) + \left(44 + \sin^2(135^\circ) + \sin^2(180^\circ)\right) \\
&+ \left(44 + \sin^2(225^\circ) + \sin^2(270^\circ)\right) + \left(44 + \sin^2(315^\circ) + \sin^2(360^\circ)\right)
\end{aligned}$$

We know that

$$\sin^2(45^\circ) = \sin^2(135^\circ) = \sin^2(225^\circ) = \sin^2(315^\circ) = \frac{1}{2},$$

and that

$$\begin{aligned} \sin^2(90°) &= 1, \\ \sin^2(180°) &= 0, \\ \sin^2(270°) &= 1, \\ \sin^2(360°) &= 0. \end{aligned}$$

Substituting those values in, we have

$$\begin{aligned} \sum_{i=1}^{360} \sin^2(i°) = &\left(44 + \sin^2(45°) + \sin^2(90°)\right) + \left(44 + \sin^2(135°) + \sin^2(180°)\right) \\ &+ \left(44 + \sin^2(225°) + \sin^2(270°)\right) + \left(44 + \sin^2(315°) + \sin^2(360°)\right) \\ = &\ 4 \cdot 44 + 4 \cdot \frac{1}{2} + 2 \cdot 1 + 2 \cdot 0 \\ = &\ 180. \end{aligned}$$