Nrich Making Waves

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1 Sketching sin(cosx)

Since $\cos x$ ranges between -1 and 1, and in this interval sin is increasing, we can say that at max $\cos x$ we get max $\sin(\cos x)$ and likewise for min. So max $\sin(\cos x) = \sin(1)$ and min $\sin(\cos x) = \sin(-1)$.

The graph crosses y axis at $x=0$, so $y = \sin(\cos 0) = \sin 1$ so it crosses at a maximum.

It crosses x axis at $y=0$ so

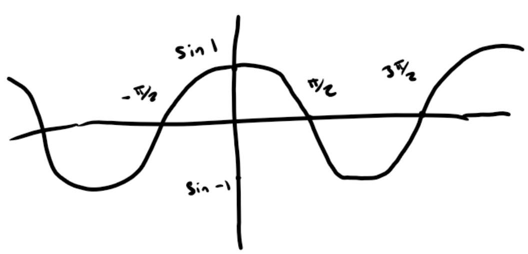
$0 = \sin(\cos x)$

$\cos x = 0, \pi, 2\pi \quad (+2\pi n)$

$x = \arccos(0, \pi, 2\pi) = \frac{\pi}{2}, \frac{3\pi}{2}$, no sol for the rest.

so $x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n$.

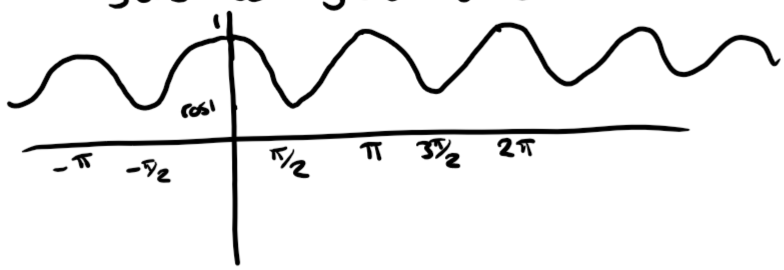
also note that if $f(x) = \sin(\cos x)$ then $f(-x) = \sin(\cos(-x)) = \sin(\cos x) = f(x)$ so f is even and has symmetry in y-axis.



2 Sketching cos(sinx)

If we do the same for $\cos(\sin x)$ we get it crosses y at 1.
Then $\sin x$ goes between -1 and 1 and in this interval $\cos x$ turns at 0.
So max is when $\sin x = 0$, $x = 0, \pi, 2\pi$
min is when $\sin x = 1$ or -1, $x = \frac{\pi}{2}, \frac{3\pi}{2}$

It cannot cross x as $0 = \cos(\sin x)$
$\sin x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n$ but $-1 \le \sin x \le 1$
so no real solutions for x.
if $f(x) = \cos(\sin x)$ then
$f(-x) = \cos(\sin(-x)) = \cos(-\sin x) = \cos(\sin x)$
$= f(x)$. So $f(x)$ is even.



3 Show that cos(sinx) > sin(cosx) for some x

Consider $\cos(\sin x) > \sin(\cos x)$

if $x=0$; $\cos(\sin 0) > \sin(\cos 0)$

$\cos 0 > \sin 1$

$\cos 0 = 1$ as sin can't be > 1 so this is true if $\sin x \neq 1$ which is only if $x = \pi/2$, not 1.

so its true

if $x=\pi$; $\cos(\sin\pi) > \sin(\cos\pi)$

$\cos 0 > \sin -1$

$\cos 0 = 1$ and by the same logic as before, the result is true.

if $x = \pi/2$; $\cos(\sin \pi/2) > \sin(\cos \pi/2)$

$\cos 1 > \sin 0 = 0$

Note $\cos x > 0$ if $0 < x < \pi/2$ and $0 < 1 < \pi/2$ so its true

if $x = \pi/4$; $\cos(\sin \pi/4) > \sin(\cos \pi/4)$

$\cos \sqrt{2}/2 > \sin \sqrt{2}/2$

Note $\cos x = \sin(\pi/2 - x)$ so

$\sin(\pi/2 - \sqrt{2}/2) > \sin \sqrt{2}/2$

$\sin x$ is increasing if $0 \leq x \leq \pi/2$

so since $\pi/2 - \sqrt{2}/2 > \sqrt{2}/2$,

$\sin(\pi/2 - \sqrt{2}/2)$ must be larger than $\sin(\sqrt{2}/2)$

4 The proof

Prove that cos(sinx) > sin(cosx)

We start a proof by contradiction as we are told what we are trying to get to so:
Assume the contrary, namely

$$sin(cosx) \geq cos(sinx)$$

Now we must note that we can only apply functions to an inequality without reversing the sign if and only if the function is increasing for the interval of the input (in this case the input is cos(sinx) and sin(cosx)). To manipulate this I wish to take the arcsin of both sides so we have to check this condition.

$$\frac{d}{dx}(arcsinx) = \frac{1}{\sqrt{1-x^2}}$$

This is always increasing my as long as it is real, which is when the input is strictly between -1 and 1. We will deal with the case when the input does not satisfy these conditions at the end so for now assume that the condition is met. Now we know the condition is true we arcsin both sides and achieve

$$arcsin(sin(cosx)) \geq arcsin(cos(sinx))$$

Now we may cancel arcsin and sin since they are inverses of each other

$$cosx \geq arcsin(cos(sinx))$$

Now, we know the identity $cosx = sin(\frac{\pi}{2} - x)$ and thus
$cos(sinx) = sin(\frac{\pi}{2} - sinx)$ Making use of this:

$$cosx \geq arcsin(sin(\frac{\pi}{2} - sinx))$$

$$cosx \geq \frac{\pi}{2} - sinx$$

$$cosx + sinx \geq \frac{\pi}{2}$$

Now intuitively we may think that cosx + sinx can never be greater than or equal to $\frac{\pi}{2}$ and this is a correct intuition. So we want to find the maximum of the function f(x) = cosx + sinx which we could do by calculus methods such as differentiation so

$$f'(x) = -sinx + cosx$$

A maximum is when f'(x) is 0 so

$$cosx - sinx = 0$$

$$cosx = sinx$$

Then divide by cosx. Note that we can do this because if cosx = 0 then sinx = 1 or -1, so the equation is not true, thus cosx is not 0.

$$tanx = 1$$

$$x = arctan1 = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

Substituting this into f(x) we get

$$f(x) = sin(\frac{\pi}{4} + n\pi) + cos(\frac{\pi}{4} + n\pi)$$

either

$$f(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

or

$$f(x) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$$

So the local maximum of f(x) = $\sqrt{2}$. Now we have to prove that this is a global maximum. We can do this by noting that cosx + sinx will oscillate between its local maximum and minimum and hence its local maximum is its global maximum. This means that cosx + sinx ≤ $\sqrt{2}$ and so cosx + sinx ≤ $\sqrt{2} < \frac{\pi}{2}$ but this contradicts what we have derived from our assumption, hence the opposite of the assumption is true so cos(sinx) > sin(cosx). Now we have to deal with the special cases, these are:

$$cos(sinx) \geq 1$$

or

$$cos(sinx) \leq -1$$

or

$$sin(cosx) \geq 1$$

or

$$sin(cosx) \leq -1$$

We can see that the last 3 of these are never possible from our graphs of the functions, so we ignore these. Also $cos(sinx) \geq 1$ is only possible if cos(sinx) = 1 since we found that the maximum of cos(sinx) = 1 when we were sketching. So we go into this case. We know this happens only when $x = n\pi, n \in \mathbb{Z}$. So we verify that the inequality is true in this case. We know, from our sketch, that sin(cosx) < 1 for all x so the inequality is trivially true. Hence the inequality is true for all cases and thus all x.

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