Tiffany Lau

Nrich: Root Tracker

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(1) The vertex of the parabola moves up and down, parallel to the y-axis (the x-coordinate does not change). The shape of the parabola remains the same.

(a) The x-intercepts are $(-1, 0)$ and $(6, 0)$

(b) $(1, 0)$ and $(4, 0)$

(c) There are no intersections with the x-axis.

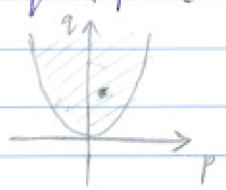
(2) The vertex moves along a parabolic curve (both the x- and y-coordinates change). The shape of the graph remains the same.

(a) The x-intercepts are $(2, 0)$ and $(8, 0)$.

(b) $(4, 0)$

(c) There are no x-intercepts.

(3) When the point (p, q) enters the region above or enclosed by the parabola $q = \frac{1}{4}p^2$ (see below), the graph of $y = x^2 + px + q$ no longer crossed the x-axis.



If the point (p, q) were below the parabola, the graph of $y = x^2 + px + q$ would cross the x-axis twice.

(4) When the roots show up on the u-axis, $q \leq \frac{1}{4}p^2$.

When the roots show up on the v-axis, $q > \frac{1}{4}p^2$.

(5) Roots: $(u, v) = (0, 1), (0, -1)$.

(6) Roots: $z_1 = 3 + 2i$, $z_2 = 3 - 2i$

(a) $z_1 + z_2 = (3 + 2i) + (3 - 2i) = 6$

(b) $z_1 \times z_2 = (3 + 2i)(3 - 2i) = 9 - 4i^2 = 13$

(c) $z_1^2 - 6z_1 + 13 = (3 + 2i)^2 - 6(3 + 2i) + 13 = 9 + 12i + 4i^2 - 18 - 12i + 13 = 0$

(d) $z_2^2 - 6z_2 + 13 = (3 - 2i)^2 - 6(3 - 2i) + 13 = 9 - 12i + 4i^2 - 18 + 12i + 13 = 0$

(7) Using the quadratic formula, the roots of $x^2+px+q=0$ are

$$z_1 = \frac{-p+\sqrt{p^2-4q}}{2}, \quad z_2 = \frac{-p-\sqrt{p^2-4q}}{2}$$

(i) When $p^2-4q \geqq 0$, z_1 and z_2 are real numbers.

The coordinates of z_1 and z_2 on the Argand diagram are

$$z_1 = \left(\frac{-p+\sqrt{p^2-4q}}{2}, 0\right), \quad z_2 = \left(\frac{-p-\sqrt{p^2-4q}}{2}, 0\right).$$

$\therefore$ They are reflections of each other in the real axis since they both lie on it.

Also, $z_1 = u+iv$ and $z_2 = u-iv$, where $v=0$ in both cases.

$$z_1+z_2 = \frac{-p+\sqrt{p^2-4q}}{2} + \frac{-p-\sqrt{p^2-4q}}{2} = \frac{-2p}{2} = -p \Rightarrow \text{This is a real number}$$

$$z_1 \times z_2 = \frac{(-p+\sqrt{p^2-4q})(-p-\sqrt{p^2-4q})}{4} = \frac{p^2-(p^2-4q)}{4} = \frac{4q}{4} = q \Rightarrow \text{This is a real number}$$

(ii) When $p^2-4q<0$, z_1 and z_2 are complex numbers.

Their coordinates on the Argand diagram are:

$$z_1 = \left(-\frac{p}{2}, \frac{\sqrt{4q-p^2}}{2}i\right), \quad z_2 = \left(-\frac{p}{2}, -\frac{\sqrt{4q-p^2}}{2}i\right)$$

This is because they can be put into the form $z_1 = u+iv$ and $z_2 = u-iv$, where $u = -\frac{p}{2}$ and $v = \frac{\sqrt{4q-p^2}}{2}$.

$\therefore$ They are reflections of each other in the real axis $(v=0)$ since they have the same u-coordinate and their v-coordinates have the same absolute value.

$$z_1+z_2 = \left(-\frac{p}{2}+\frac{\sqrt{4q-p^2}}{2}i\right) + \left(-\frac{p}{2}-\frac{\sqrt{4q-p^2}}{2}i\right) = -\frac{2p}{2} = -p \Rightarrow \text{A real number}$$

$$z_1 \times z_2 = \left(-\frac{p}{2}\right)^2 - \left(\frac{\sqrt{4q-p^2}}{2}i\right)^2 = \frac{p^2}{4} + \frac{4q-p^2}{4} = \frac{4q}{4} = q \Rightarrow \text{A real number}$$