Solutions to the first four parts (Questions 1 and 2)

1. Let $n = 2x + 1$ where $x \in \mathbb{Z}$

$$9^n + 1^n = 9^{2x+1} + 1^{2x+1}$$
$$= (9^2)^x \times 9 + 1$$

$$9 \equiv 9 \bmod 10$$

$$9^2 \equiv 81 \bmod 10$$
$$\equiv 1 \bmod 10$$

$$1 \equiv 1 \bmod 10$$

$$\therefore 9^n + 1^n \equiv (1 \bmod 10)^x \times 9 \bmod 10 + 1 \bmod 10$$
$$\equiv (1 \times 9 + 1) \bmod 10$$
$$\equiv 10 \bmod 10$$
$$\equiv 0 \bmod 10$$

QED

2. Let $n = 2x+1$, $x \in \mathbb{Z}$

$$7^n + 3^n = 7^{2x+1} + 3^{2x+1}$$
$$= (7^2)^x \times 7 + (3^2)^x \times 3$$

$7 \equiv 7 \bmod 10$

$7^2 \equiv 49 \bmod 10$
$\equiv 9 \bmod 10$

$3 \equiv 3 \bmod 10$

$3^2 \equiv 9 \bmod 10$

so $7^n + 3^n \equiv (9 \bmod 10)^x \times 7 \bmod 10 + (9 \bmod 10)^x \times 3 \bmod 10$
$\equiv (9^x \times 7) \bmod 10 + (9^x \times 3) \bmod 10$
$\equiv (10 \times 9^x) \bmod 10$

$10z \equiv 0 \bmod 10$, $z \in \mathbb{Z}$

$\therefore 7^n + 3^n \equiv 0 \bmod 10$

QED

3. Let $n = 2x$, $x \in \mathbb{Z}$

$$8^n - 2^n = 8^{2x} - 2^{2x}$$
$$= (8^2)^x - (2^2)^x$$

$$8^2 \equiv 64 \bmod 10$$
$$\equiv 4 \bmod 10$$
$$2^2 \equiv 4 \bmod 10$$

$$\text{So } 8^n - 2^n \equiv (4 \bmod 10)^x - (4 \bmod 10)^x$$
$$\equiv 4^x \bmod 10 - 4^x \bmod 10$$
$$\equiv (4^x - 4^x) \bmod 10$$
$$\equiv 0 \bmod 10$$

QED

4. Let $n = 2x$, $x \in \mathbb{Z}$

$$6^n - 4^n = 6^{2x} - 4^{2x}$$
$$= (6^2)^x - (4^2)^x$$

$$6^2 \equiv 36 \bmod 10$$
$$\equiv 6 \bmod 10$$
$$4^2 \equiv 16 \bmod 10$$
$$\equiv 6 \bmod 10$$

So $6^n - 4^n \equiv (6 \bmod 10)^x - (6 \bmod 10)^x$
$$\equiv 6^x \bmod 10 - 6^x \bmod 10$$
$$\equiv (6^x - 6^x) \bmod 10$$
$$\equiv 0 \bmod 10$$

QED