

Solutions to the first four parts (Questions 1 and 2)

1. Let $n = 2x + 1$ Where $x \in \mathbb{Z}$

$$\begin{aligned}9^n + 1^n &= 9^{2x+1} + 1^{2x+1} \\ &= (9^2)^x \times 9 + 1\end{aligned}$$

$$9 \equiv 9 \pmod{10}$$

$$\begin{aligned}9^2 &\equiv 81 \pmod{10} \\ &\equiv 1 \pmod{10}\end{aligned}$$

$$1 \equiv 1 \pmod{10}$$

$$\begin{aligned}\therefore 9^n + 1^n &\equiv (1 \pmod{10})^x \times 9 \pmod{10} + 1 \pmod{10} \\ &\equiv (1 \times 9 + 1) \pmod{10} \\ &\equiv 10 \pmod{10} \\ &\equiv 0 \pmod{10}\end{aligned}$$

QED

2. Let $n = 2x+1$, $x \in \mathbb{Z}$

$$\begin{aligned} 7^n + 3^n &= 7^{2x+1} + 3^{2x+1} \\ &= (7^2)^x \times 7 + (3^2)^x \times 3 \end{aligned}$$

$$7 \equiv 7 \pmod{10}$$

$$\begin{aligned} 7^2 &\equiv 49 \pmod{10} \\ &\equiv 9 \pmod{10} \end{aligned}$$

$$3 \equiv 3 \pmod{10}$$

$$3^2 \equiv 9 \pmod{10}$$

$$\begin{aligned} \text{so } 7^n + 3^n &\equiv (9 \pmod{10})^x \times 7 \pmod{10} + (9 \pmod{10})^x \times 3 \pmod{10} \\ &\equiv (9^x \times 7) \pmod{10} + (9^{2x} \times 3) \pmod{10} \\ &\equiv (10 \times 9^{2x}) \pmod{10} \end{aligned}$$

$$10z \equiv 0 \pmod{10}, \quad z \in \mathbb{Z}$$

$$\therefore 7^n + 3^n \equiv 0 \pmod{10}$$

QED

3. Let $n = 2^{2x}$, $x \in \mathbb{Z}$

$$\begin{aligned}8^n - 2^n &= 8^{2^x} - 2^{2^x} \\ &= (8^2)^x - (2^2)^x\end{aligned}$$

$$8^2 \equiv 64 \pmod{10}$$

$$\equiv 4 \pmod{10}$$

$$2^2 \equiv 4 \pmod{10}$$

$$\text{so } 8^n - 2^n \equiv (4 \pmod{10})^{2^x} - (4 \pmod{10})^{2^x}$$

$$\equiv 4^{2^x} \pmod{10} - 4^{2^x} \pmod{10}$$

$$\equiv (4^{2^x} - 4^{2^x}) \pmod{10}$$

$$\equiv 0 \pmod{10}$$

QED

4. Let $n = 2x$, $x \in \mathbb{Z}$

$$\begin{aligned}6^n - 4^n &= 6^{2x} - 4^{2x} \\ &= (6^2)^x - (4^2)^x\end{aligned}$$

$$6^2 \equiv 36 \pmod{10}$$

$$\equiv 6 \pmod{10}$$

$$4^2 \equiv 16 \pmod{10}$$

$$\equiv 6 \pmod{10}$$

$$\text{So } 6^n - 4^n \equiv (6 \pmod{10})^x - (6 \pmod{10})^x$$

$$\equiv 6^x \pmod{10} - 6^x \pmod{10}$$

$$\equiv (6^x - 6^x) \pmod{10}$$

$$\equiv 0 \pmod{10}$$

QED